Analog & Digital Communication Systems

(5th Semester E&TC)

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Communication System

Energy of a signal:

$x(t)$
(fun. of time)

Energy of $x(t)$, $E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$

Example: $x(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{cases} = e^{-t} u(t).$

$$E_x = \int_0^{\infty} e^{-2t} \, dt = \frac{e^{-2t}}{-2} \Big|_0^{\infty} = \frac{1}{2}[1-0] = \frac{1}{2}$$

If $E_x < \infty$, then $x(t)$ is termed as an energy signal.

Power of a signal:

Power of $x(t)$, $P_x = \lim_{\tilde{T} \to \infty} \frac{1}{\tilde{T}} \underbrace{\int_{\tilde{T}} |x(t)|^2 dt}_{\text{Energy in a window of size } \tilde{T}}$

$$= \lim_{\tilde{T} \to \infty} \frac{\text{Energy in a window of size } \tilde{T}}{\tilde{T}}$$

If $P_x < \infty$, then $x(t)$ is termed as a power signal.

If $x(t)$ is an Energy signal.

$$P_x = \lim_{\tilde{T} \to \infty} \frac{1}{\tilde{T}} \int_{\tilde{T}} |x(t)|^2 dt$$

$$\leq \lim_{\tilde{T} \to \infty} \frac{1}{\tilde{T}} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \lim_{\tilde{T} \to \infty} \frac{E_x}{\tilde{T}} = 0$$

$\Rightarrow P_x \leq 0$

$\Rightarrow P_x = 0$ ($\because P_x$ is a +ve qty)

Power of an Energy Signal is 0.

If $x(t)$ is a power signal, Energy in a window of size $\tilde{T}$

$$\simeq P_x \tilde{T}$$

$\Rightarrow$ Total energy $= \lim\limits_{\tilde{T}\to\infty}$ Energy in a window of size $\tilde{T}$

$$= \lim_{\tilde{T}\to\infty} P_x \tilde{T} = \infty$$

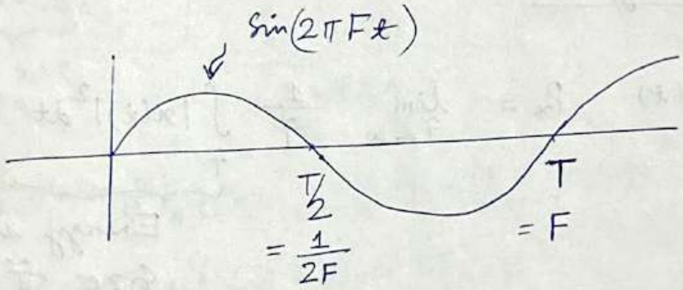
$\therefore$ Energy of a power signal is ∞.

Periodic signals are power signals.

Periodic signals

$x(t)$ is periodic if

$$x(t) = x(t + kT), \quad \forall t, k \in \mathbb{Z}$$



$F \to$ frequency

$T = \frac{1}{F}$ (Fundamental Period)

$$\sin(2\pi F[t + kT]) = \sin(2\pi Ft + 2\pi FkT) = \sin(2\pi Ft + 2\pi k)$$

$$= \sin(2\pi Ft).$$

$A \sin(2\pi Ft + \phi)$: Periodic with Period $= \frac{1}{F}$

A $\downarrow$ Amplitude; ϕ $\uparrow$ Phase

Power of a periodic signal:

Let T be the period of periodic signal.

$$P_x = \lim_{\tilde{T}\to\infty} \frac{1}{\tilde{T}} \int_{\tilde{T}} |x(t)|^2 dt$$

Chose $\tilde{T} = mT$

$$m\to\infty \Rightarrow mT \to \infty$$
$$\Rightarrow \tilde{T} \to \infty$$

$$P_x = \lim_{m\to\infty} \frac{1}{mT} \underbrace{\int_{-\frac{mT}{2}}^{\frac{mT}{2}} |x(t)|^2 dt}$$

Energy in m periods = m Energy in a period

$$= \lim_{m\to\infty} \frac{1}{mT}\, m \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$\boxed{P_x = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt}$$

$$= \frac{\text{Energy in a window of size } T}{T}$$

$T \rightsquigarrow$ Period.

Example:

$$x(t) = A\cos(2\pi F t), \quad T = \frac{1}{F}$$

$$\text{Power, } P = \frac{1}{T}\int_{-T/2}^{T/2} A^2\cos^2(2\pi F t)\, dt$$

$$\frac{1}{T}\times 2\int_0^{T/2} A^2\cos^2(2\pi F t)\, dt$$

$$= \frac{2A^2}{T}\int_0^{T/2} \frac{1+\cos(4\pi F t)}{2} dt$$

$$= \frac{A^2}{T}\int_0^{T/2} 1+\cos(4\pi F t)\, dt$$

$$= \frac{A^2}{T}\left\{\frac{T}{2} + \frac{\sin(4\pi F t)}{4\pi F}\Big|_0^{T/2}\right\}$$

$$= \frac{A^2}{2}$$

$$= \frac{A^2}{T}\int_{-T/2}^{T/2} \frac{1+\cos(4\pi F t)}{2} dt$$

$$= \frac{A^2}{T}\left\{\frac{1}{2}\cdot T + \frac{1}{2}\cdot \frac{\sin 4\pi F t}{4\pi F}\Big|_{-T/2}^{T/2}\right\}$$

$$= \frac{A^2}{2} + \frac{A^2}{2}\frac{1}{8\pi F}\cdot 0$$

$$= \frac{A^2}{2}$$

Power of $A\cos(2\pi F t + \phi)$ is $\frac{A^2}{2}$.

{ Power depends on Amplitude, independent of Phase)

Frequency Domain Representation of Signals

One of the fundamental tools used in comm. system.

→ Spectrum

Fourier Series

||

Discrete in frequency

Fourier Transform

||

Continuous in Frequency.

Fourier Series

Defined for a periodic signal. $x(t)$

$x(t)$: Periodic with period T.

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi k F_0 t}$$

Linear Combination

$F_0 = \frac{1}{T}$: Fundamental Frequency of $x(t)$.

C_k : k^{th} F.S. coefficient (or) Coefficient of k^{th} Harmonic.

$$e^{j2\pi k F_0 t} = \underbrace{\cos(2\pi k F_0 t) + j \sin 2\pi k F_0 t}_{\text{complex sinusoid.}} \rightarrow K^{th} \text{ Harmonic}$$

$kF_0 \rightarrow$ Multiple of Fundamental frequency F_0

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi l F_0 t}\, dt = C_l$$

$$= \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_{k=-\infty}^{\infty} C_k e^{j2\pi k F_0 t}\right) e^{-j2\pi l F_0 t}\, dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} C_k \cdot \frac{1}{T}\cdot \int_{-T/2}^{T/2} e^{j2\pi(k-l)F_0 t}\, dt$$

$$\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-l)F_0 t}\, dt$$

$$= \begin{cases} \frac{1}{T}\cdot T = 1 & k = l \\ 0 & k \neq l \end{cases}$$

$$\frac{1}{T}\left.\frac{e^{j2\pi(k-l)F_0 t}}{j2\pi(k-l)F_0}\right|_{-T/2}^{T/2}$$

$$\frac{1}{T}\,\frac{1}{j2\pi(k-l)F_0}\underbrace{\left\{e^{j\pi(k-l)} - e^{-j\pi(k-l)}\right\}}$$

$$= \frac{1}{T}\,\frac{1}{2\pi(k-l)F_0}\times 0$$

$\Delta\phi$ = Phase Difference $= 2\pi(k-l)$

$$= 0.$$

$\cancel{\cos\pi(k-l)} + j\sin\pi(k-l)$

$-\cancel{\cos\pi(k-l)} + j\sin(k-l)\pi$

$= j2\sin\pi(k-l)$

$= 0,\ (\because k \neq l)$

$$\underbrace{e^{j2\pi k F_0 t}\quad e^{j2\pi l F_0 t}}$$ $[l, k \in Z]$

Different Harmonics are orthogonal.

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)\cdot y^*(t)\, dt = 0$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi F_0 k t}\cdot\left(e^{j2\pi l F_0 t}\right)^* dt = 0.$$

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi l F_0 t}\, dt = \sum_{k=-\infty}^{\infty} C_k \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} e^{j2\pi(k-l)F_0 t}\, dt}_{= \begin{cases} 1, & k=l \\ 0, & k\neq l \end{cases}}$$

$$= \sum_{k=-\infty}^{\infty} C_k\, \delta(k-l)$$

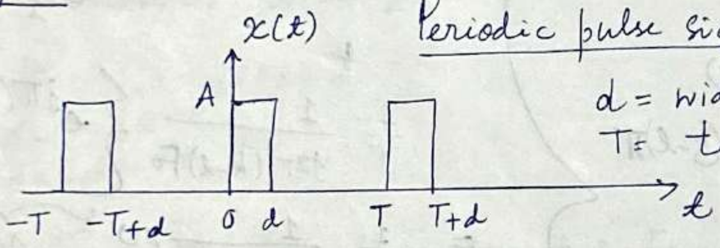
$$= C_k .$$

$$\frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{-j2\pi l F_0 t} = C_l .$$

Coefficient of l^{th} Harmonic.

<u>Example</u>

<u>Periodic pulse signal</u>



d = width of pulse

T = time period

$$d = \frac{T}{4}$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{j2\pi k F_0 t} \quad , \quad F_0 = \frac{1}{T}$$

<u>$k, l \in Z$</u>

$$C_l = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j2\pi l F_0 t}\, dt$$

(over a period) $\Rightarrow$ <u>t to $t+T$</u>

$$C_l = \frac{1}{T}\int_0^T x(t)\, e^{-j2\pi l F_0 t}\, dt \qquad C_0 = \frac{1}{T}\int_0^{T/4} A\, dt$$

$$= \frac{1}{T}\int_0^{T/4} A \cdot e^{-j2\pi l F_0 t}\, dt \qquad = \frac{1}{T} \times A \times \frac{T}{4} = \frac{A}{4}$$

$$= \frac{A}{T} \left. \frac{e^{-j2\pi l F_0 t}}{-j2\pi l F_0} \right|_0^{T/4} \qquad \Rightarrow \boxed{C_0 = \frac{A}{4}} \leftarrow \text{DC coefficient.}$$

$$= \frac{A}{2\pi l j}\left[1 - e^{-j\frac{2\pi l}{4}}\right]$$

$$\Rightarrow C_l = \frac{A}{2\pi l j} e^{-j\frac{\pi l}{4}}\left[e^{+j\frac{\pi l}{4}} - e^{-j\frac{\pi l}{4}}\right]$$

$$\Rightarrow \boxed{C_l = \frac{A}{\pi l} e^{-j\frac{\pi l}{4}} \sin\left[\frac{\pi l}{4}\right]}$$

l^{th} F.S. coefficient ($l \neq 0$)

$$\therefore \boxed{C_l = \begin{cases} \frac{A}{4}, & l = 0 \\ \frac{A}{\pi l} e^{-j\frac{\pi l}{4}} \sin\left(\frac{\pi l}{4}\right), & l \neq 0 \end{cases}}$$

$$|C_l| = \left| \frac{A}{\pi l} \cdot e^{-j\frac{\pi l}{4}} \sin\left(\frac{\pi l}{4}\right) \right|$$

$$\Rightarrow |C_l| = \left| \frac{A}{\pi l} \sin\left(\frac{\pi l}{4}\right) \right| \quad \text{: Magnitude Spectrum.}$$

Power of Periodic Signal:

For a periodic signal $x(t)$,

$$P_x = \lim \frac{1}{T}\int_T |x(t)|^2\, dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{j2\pi k F_0 t}$$

$$x^*(t) = \sum_{m=-\infty}^{\infty} C_m^*\, e^{-j2\pi m F_0 t}$$

$$\text{Power} = P_x = \frac{1}{T}\int_0^T x(t)\cdot x^*(t)\,dt$$

$$= \frac{1}{T}\int_0^T \left(\sum_{k=-\infty}^{\infty} C_k e^{j2\pi kF_0 t}\right)\left(\sum_{m=-\infty}^{\infty} C_m^* e^{-j2\pi mF_0 t}\right)dt$$

$$= \frac{1}{T}\int_0^T \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_k C_m^* e^{+j2\pi(k-m)F_0 t}\,dt$$

$$= \cancel{\tfrac{1}{T}}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_k\cdot C_m^* \underbrace{\frac{1}{T}\int_0^T e^{j2\pi(k-m)F_0 t}\,dt}$$

$$= \delta(k-m)$$

$$= \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_k C_m^* \,\delta(k-m)$$

$$\boxed{P_x = \sum_{k=-\infty}^{\infty} C_k\cdot C_k^* = \sum_{k=-\infty}^{\infty} |C_k|^2}$$

$$\boxed{\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2}$$

Power in terms of spectrum.

Parseval's Theorem.

Fourier Transform (F.T.)

Spectrum of an aperiodic continuous time Signal $x(t)$

Consider an aperiodic signal $x(t)$. Its Fourier Transform is,

$$\boxed{X(F) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi F t}\, dt}$$

F = Frequency

$$\underbrace{\underset{\text{Time Domain}}{x(t)} \longleftrightarrow \underset{\text{Frequency Domain}}{X(F)}}_{\text{Fourier Transform Pair.}}$$

$X(F) \rightarrow$ Complex

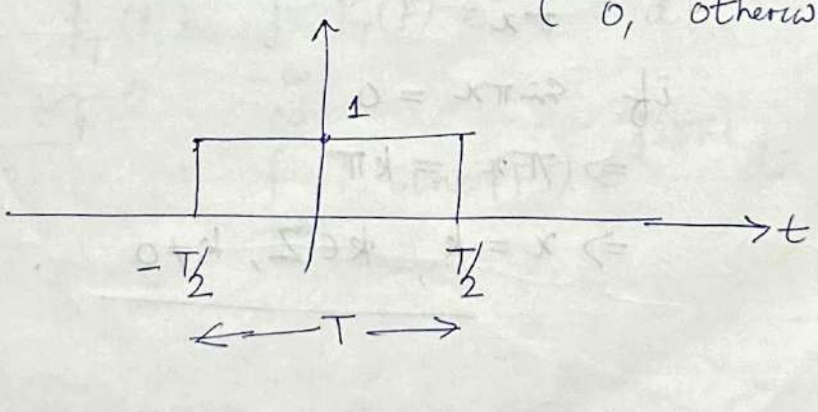
- $\rightarrow |X(F)| \rightarrow$ Magnitude Spectrum.
- $\rightarrow \angle X(F) \rightarrow$ Phase Spectrum.

Given $X(F)$,

IFT :
$$\boxed{x(t) = \int_{-\infty}^{\infty} X(F) \cdot e^{j2\pi F t}\, dF}$$

Example :-

$$x(t) = p_T(t) = \begin{cases} 1, & |t| \leq T/2 \\ 0, & \text{otherwise} \end{cases}$$



$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi F t}\, dt$$

$$P_T(F) = \int_{-T/2}^{T/2} 1 \cdot e^{-j2\pi F t}\, dt$$

$$= 2 \int_{0}^{T/2} \cos(2\pi F t)\, dt$$

$$= 2 \left. \frac{\sin(2\pi F t)}{2\pi F} \right|_{0}^{T/2}$$

$$\boxed{P_T(F) = \frac{\sin(\pi F T)}{\pi F}}$$

$$= T\, \frac{\sin(\pi F T)}{\pi F T}$$

$$\boxed{P_T(F) = T\, \mathrm{sinc}(FT)}$$

sinc function

$$\mathrm{sinc}\, x = \frac{\sin \pi x}{\pi x}$$

$$\lim_{x \to 0} \mathrm{sinc}(x) = \lim_{x \to 0} \frac{\sin \pi x}{\pi x} = 1$$

$$\mathrm{sinc}\, x = \frac{\sin \pi x}{\pi x} = 0$$

if $\sin \pi x = 0$

$$\Rightarrow \pi x = k\pi$$

$$\Rightarrow x = k, \quad k \in \mathbb{Z},\ k \neq 0$$

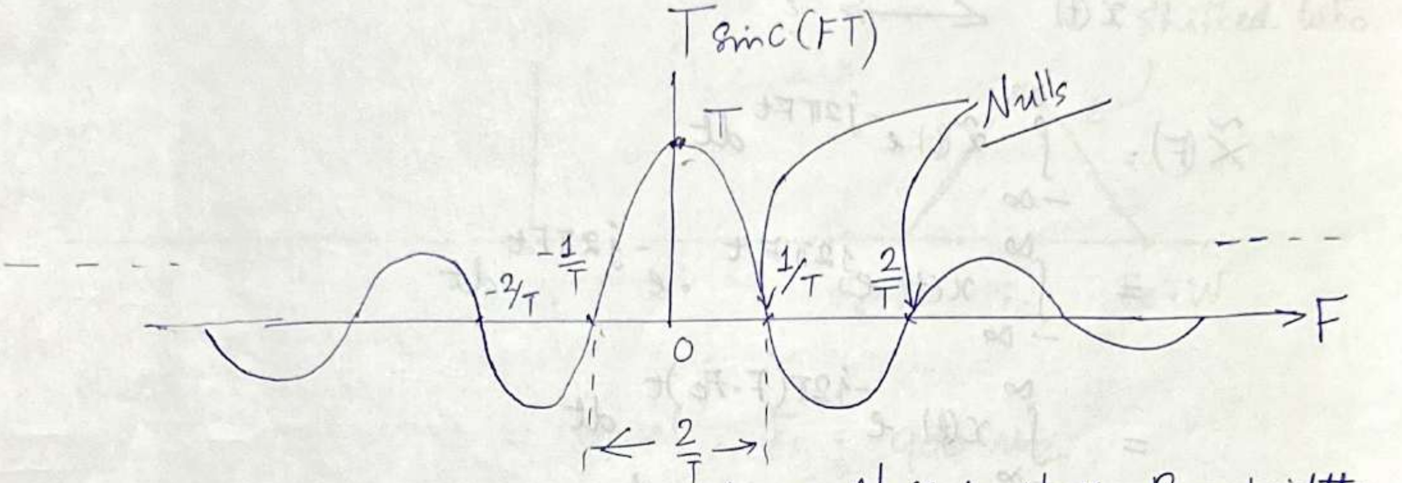
$P_T(F) = T\,\mathrm{sinc}(FT)$

At $F=0$, $T\,\mathrm{sinc}(0) = T$

At $F = \frac{k}{T}$, $T\,\mathrm{sinc}(k) = 0$

$$T\,\mathrm{sinc}\,FT = \frac{T \sin \pi F T}{\pi F T}$$

Looks like $\sin(x)$ except amplitude is decreasing due to $\frac{1}{F}$.



$$\boxed{p_T(t) \longleftrightarrow P_T(F) = T\,\mathrm{sinc}(FT)}$$

$$p_T(t) = \int_{-\infty}^{\infty} P_T(F)\, e^{j2\pi F t}\, dF$$

Time Domain Pulse.

$$= \int_{-\infty}^{\infty} T\,\mathrm{sinc}(FT)\, e^{j2\pi F t}\, dF$$

Modulation Property of F.T. :

$$\underbrace{\tilde{x}(t)}_{\text{Modulated Signal.}} = \underbrace{x(t)}_{\text{Signal}} \, \underbrace{e^{j2\pi F_c t}}_{\text{complex Sinusoid}}$$

Carrier

F_c = Carrier Frequency

$$x(t) \longleftrightarrow X(F)$$

$$\tilde{x}(t) \longleftrightarrow ?$$

$$\tilde{X}(F) = \int_{-\infty}^{\infty} \tilde{x}(t)\, e^{-j2\pi F t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{j2\pi F_c t} \cdot e^{-j2\pi F t}\, dt$$

$$= \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(F-F_c)t}\, dt}_{X(F-F_c)}$$

$$\boxed{\tilde{X}(F) = X(F-F_c)}$$

F.T. of Modulated Signal

Fourier Transform ~~of x(t), ie,~~ X(F) shifted by F_c.

Modulation in Time = Shift in frequency by F_c

Carrier Frequency

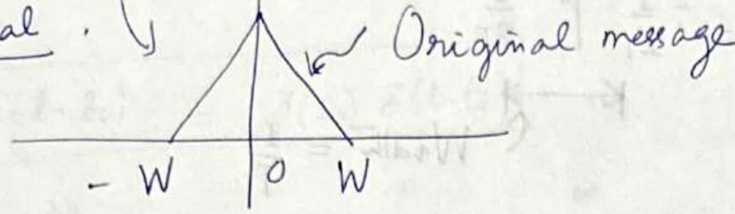
Example :

GSM Comm. System.

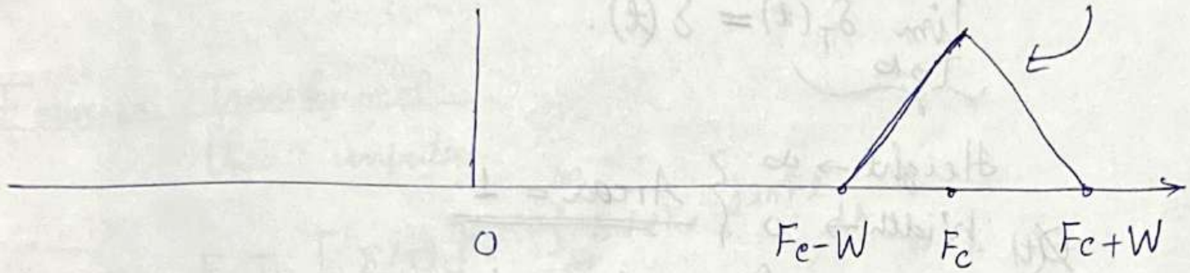
Global System fore Mobile

Dominant Wireless Technology.

Baseband Signal.



$= 100\ KHz$



Typical, $F_c = 900\ MHz$ (or) $1800\ MHz$

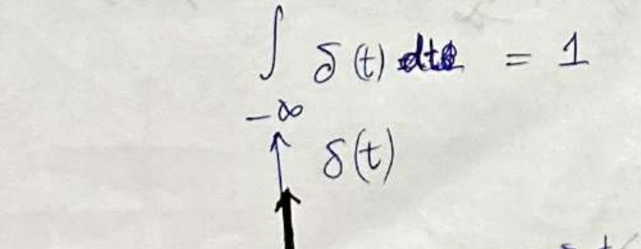
$$\boxed{F_c \gg \text{Message Bandwidth } (W)}$$

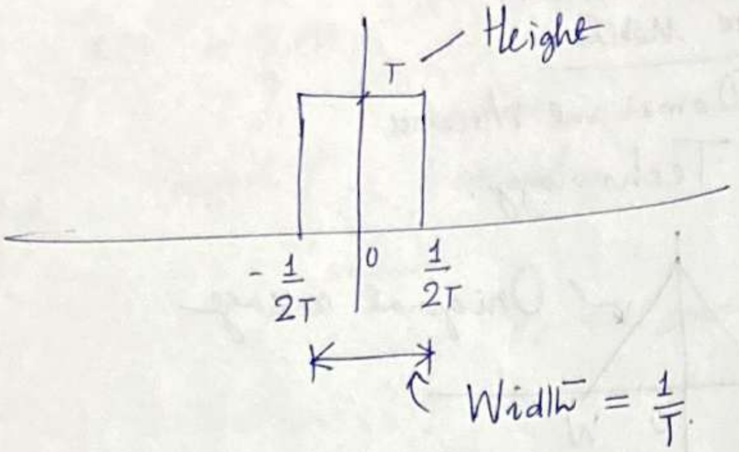
Unit impulse Function :

Dirac Delta Function.

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{Undefined}, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$





Area $\delta_T(t) = T \times \frac{1}{T} = 1$

$$\underbrace{\lim_{T \to \infty}}\ \delta_T(t) = \delta(t).$$

$$\left.\begin{array}{l} \text{Height} \to \infty \\ \text{Width} \to 0 \end{array}\right\} \underline{\text{Area} = 1}$$

For any $\phi(t)$ continuous ~~fun ϕ_t~~ at 0,

$$\boxed{\phi(t)\,\delta(t) = \phi(0)\,\delta(t)}$$

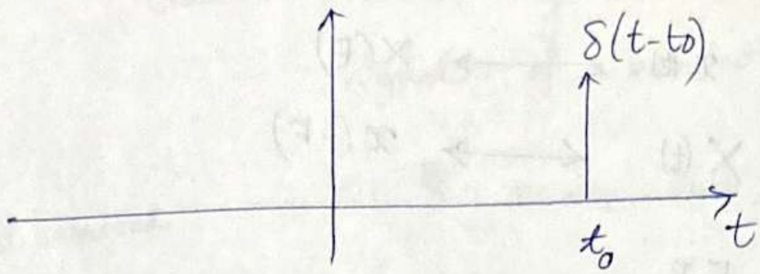
This operation picks value of $\phi(t)$ at $t=0$

Further, : $\left.\phi(t)\right|_{t=0}$

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt = \int_{-\infty}^{\infty} \phi(0)\,\delta(t)\,dt$$

$$= \phi(0) \cdot \int_{-\infty}^{\infty} \delta(t)\,dt = \phi(0)$$

$$\delta(t-t_0) = 0, \text{ if } t \neq t_0$$



$$\boxed{\phi(t)\,\delta(t-t_0) = \phi(t_0)\,\delta(t-t_0)}$$

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{\infty} \phi(t_0)\,\delta(t-t_0)\,dt = \phi(t_0)$$

Fourier Transform of

Unit impulse.

$$\text{F.T.}\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi Ft}\,dt = 1$$

$$\boxed{\delta(t) \longleftrightarrow 1, \; -\infty < F < \infty}$$

$$\text{F.T.}\left[\delta(t-t_0)\right] = \int_{-\infty}^{\infty} \delta(t-t_0)\,.\, e^{-j2\pi Ft}\,dt$$

$$= e^{-j2\pi Ft_0}$$

$$\boxed{\delta(t-t_0) \longleftrightarrow e^{-j2\pi Ft_0}}$$

Shifted impulse — Complex sinusoid.

Duality Property of F.T. :

$$x(t) \longleftrightarrow X(F)$$

$$X(t) \longleftrightarrow x(-F)$$

From Inverse F.T.,

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi F t} dF$$

Interchange t & F

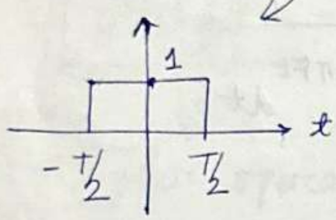
$$x(F) = \int_{-\infty}^{\infty} X(t) . e^{j2\pi F t} dt$$

$$\Rightarrow x(-F) = \int_{-\infty}^{\infty} X(t) . e^{-j2\pi F t} dt = FT\left[X(t)\right]$$

Example :-

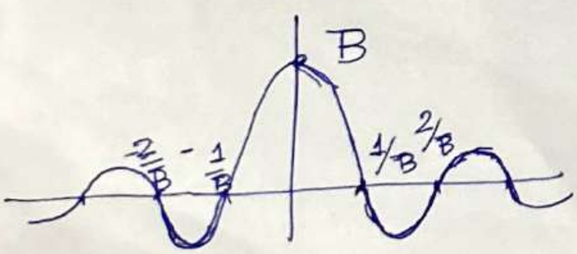
$$\underset{x(t)}{p_T(t)} \longleftrightarrow \underset{X(F)}{T \operatorname{sinc}(FT)}$$

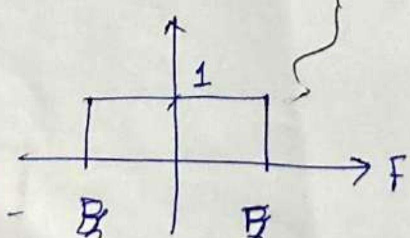
$$= T \frac{\sin(\pi F T)}{\pi F T}$$



$$= \begin{cases} 1, & |t| \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

$$B \operatorname{sinc}(Bt) \longleftrightarrow p_B(-F)$$





Window in freqeuency

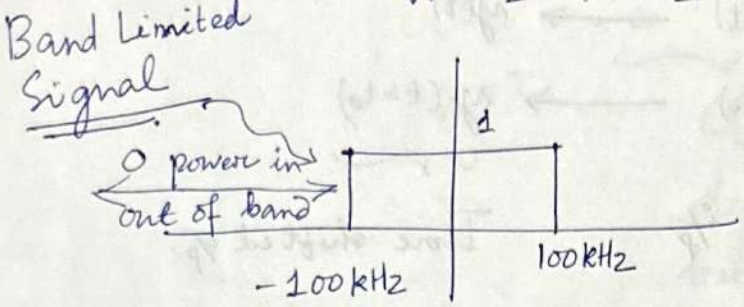
Very important in comm. system

To generate Band Limited Signal.

Example

GSM: Need to generate a band limited signal of bandwidth = 100 kHz

$$W = \frac{B}{2} = 100\text{ kHz} \Rightarrow B = 200\text{ kHz}$$



$$B\,\text{sinc}(Bt) = 200\text{ kHz}\ \text{sinc}(200\text{ kHz}\ t)$$

GSM → Band Limited Signal

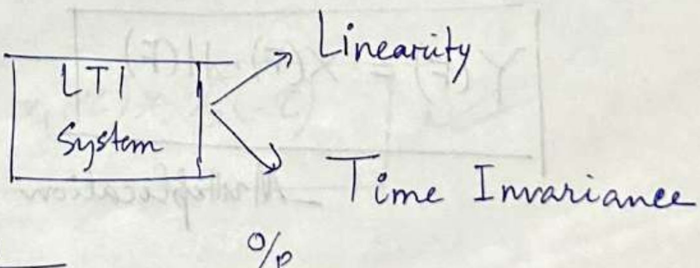
$$B/2 = 100\text{ kHz}$$

$$200\text{ kHz}\ \text{sinc}(200\text{ kHz}\ t) \longleftrightarrow \underbrace{P_{200\text{kHz}}(F)}.$$

Band Limited Signal

TRANSMISSION of a signal through a Linear System:

Consider a signal $x(t)$ given as an input to an LTI system.



i/p	o/p
$x_1(t) \longrightarrow$	$y_1(t)$
$x_2(t) \longrightarrow$	$y_2(t)$

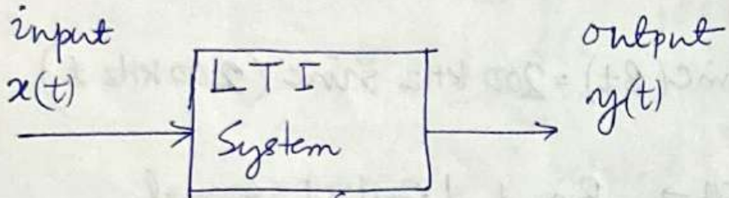
$$a x_1(t) + b x_2(t) \longrightarrow a y_1(t) + b y_2(t).$$

Linear combination of i/p s produce same linear combination of corresponding o/p s.

TIME INVARIANCE

$$x(t) \longrightarrow y(t)$$

$$\underbrace{x(t-t_0)}_{\text{Time shifted i/p}} \longrightarrow \underbrace{y(t-t_0)}_{\text{Time shifted o/p}}$$



Impulse Response

$h(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Convolution operator.

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau .$$

$$\boxed{Y(F) = X(F) \cdot H(F).}$$

Multiplication in Frequency Domain.

Cross-Correlation

Consider $x_1(t)$, ~~y(t)~~ $x_2(t)$

Cross-correlation function

$$\underbrace{r_{12}(\tau)}_{\text{Lag}=\tau} = \underbrace{\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt}$$

Characterizes the extent/degree of similarity between $x_1(t)$, $x_2(t)$ for a shift of τ.

$$r_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*[-(\tau-t)]\, dt$$

$$\tilde{x}_2(t) = x_2(-t)$$

$$\tilde{x}_2(\tau-t) = x_2(t-\tau)$$

$$\Rightarrow \tilde{x}_2^*(\tau-t) = x_2^*(t-\tau)$$

$$= \int_{-\infty}^{\infty} x_1(t)\, \tilde{x}_2^*(\tau-t)\, dt$$

$$= x_1(t) * \tilde{x}_2(t)$$

$$= x_1(t) * x_2(-t)$$

$$\boxed{r_{12}(\tau) = x_1(\tau) * x_2^*(-\tau)}$$

$$r_{12}(\tau) = x_1(\tau) * x_2^*(-\tau)$$

$$\downarrow$$

$$S_{12}(F) = X_1(F) \cdot X_2^*(F)$$

$$FT\left[x_2^*(-t) \right]$$

$$x_2(t) \Longleftrightarrow X_2(F)$$

$$X_2(F) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi Ft}\, dt$$

$$X_2^*(F) = \int_{-\infty}^{\infty} x_2^*(t)\, e^{j2\pi Ft}\, dt$$

$$\tilde{t} = -t$$

$$\Rightarrow d\tilde{t} = -dt$$

$$X_2^*(F) = \int_{\infty}^{-\infty} x_2^*(-\tilde{t})\, e^{-j2\pi F\tilde{t}}\, (-d\tilde{t})$$

$$= \int_{-\infty}^{\infty} x_2^*(-\tilde{t})\, e^{-j2\pi F\tilde{t}}\, d\tilde{t}$$

$$\boxed{\underbrace{x_1(\tau) * x_2^*(-\tau)}_{r_{12}(\tau)} \Longleftrightarrow \underbrace{X_1(F) \cdot X_2^*(F)}_{S_{12}(F)}}$$

Cross-correlation

AUTO-CORRELATION

$$x_1(\tau) * x_2(-\tau) \longleftrightarrow X_1(F) \cdot X_2^*(F)$$

$$r_{11}(\tau) = x_1(t) * x_1^*(-t)$$

$$S_{11}(F) = X_1(F) \cdot X_1^*(F)$$

$$\boxed{S_{11}(F) = |X_1(F)|^2}$$

Consider, $x(t)$

Autocorrelation, $R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t-\tau)\, dt$

$$= x(\tau) * x^*(-\tau)$$

$$\boxed{S_{xx}(F) = X(F)\, X^*(F) = |X(F)|^2}$$

F.T. of Auto-correlation.

$$R_{xx}(\tau) \longleftrightarrow S_{xx}(F)$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(F)\, e^{j2\pi F\tau}\, dF$$

$$R_{xx}(0) = \int_{-\infty}^{\infty} S_{xx}(F) \cdot dF$$

$$\boxed{R_{xx}(0) = \int_{-\infty}^{\infty} |X(F)|^2 dF}$$

Also, $R_{xx}(0) = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$

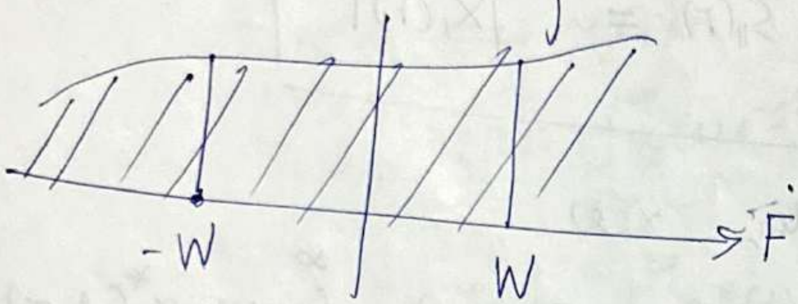
$$= \underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\text{Energy of } x(t)}$$

$$\boxed{R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(F)|^2\, dF} \qquad S_{xx}(F) = |X(F)|^2$$

Parseval's Relation.

Total Energy of the signal $x(t)$

$|X(F)|^2 = S_{xx}(F)$



Energy in the band $[-W, W]$

$$= \int_{-W}^{W} S_{xx}(F)\, dF = \int_{-W}^{W} |X(F)|^2\, dF$$

$S_{xx}(F) = |X(F)|^2$: Energy Spread or Distribution of energy of $x(t)$ in frequency Domain.

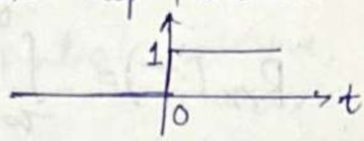
Energy Spectral Density (ESD) of $x(t)$.

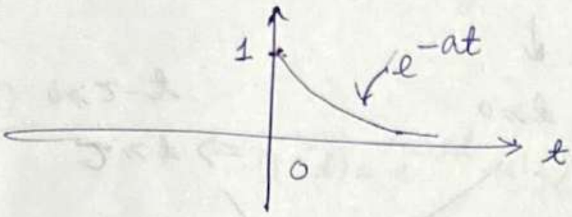
Example:
(Auto-correlation)

$$x(t) = e^{-at} u(t). \qquad u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$= \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$\uparrow$ unit step function





Auto-correlation of $e^{-at} u(t)$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x(t-\tau)\, dt \quad [\text{For real } x(t)]$$

$$= \int_{-\infty}^{\infty} e^{-at} \underbrace{u(t)}_{t \geq 0} \cdot e^{-a(t-\tau)} \underbrace{u(t-\tau)}_{t \geq \tau}\, dt$$

$\cancel{t} \geq \tau$

$$= \int_{\tau}^{\infty} e^{-at} \cdot e^{-a(t-\tau)}\, dt$$

$$= \int_{\tau}^{\infty} e^{-2at} e^{a\tau}\, dt$$

$$= e^{a\tau} \left. \frac{e^{-2at}}{-2a} \right|_{\tau}^{\infty}$$

$$= \frac{e^{a\tau}}{2a} \left[e^{-2a\tau} - 0 \right] = \frac{e^{-a\tau}}{2a}.$$

$$\boxed{R_{xx}(\tau) = \frac{e^{-a\tau}}{2a}} \quad [\tau \geq 0]$$

<u>If $\tau \leq 0$</u>

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} e^{-at} u(t) \cdot e^{-a(t-\tau)} u(t-\tau)\, dt$$

$t \geq 0$

$t - \tau \geq 0$
$\Rightarrow t \geq \tau$

$t \geq 0$ $(\because \tau \text{ is -ve})$

$$R_{xx}(\tau) = \int_{0}^{\infty} e^{-at} \cdot e^{-a(t-\tau)}\, dt$$

$$= \int_0^{\infty} e^{-2at}\, e^{a\tau}\, dt$$

$$= e^{a\tau} \left. \frac{e^{-2at}}{-2a} \right|_0^{\infty}$$

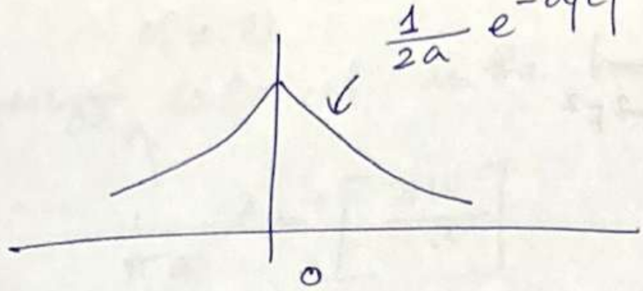
$$= \frac{e^{a\tau}}{2a} [1 - 0]$$

$$= \frac{e^{a\tau}}{2a},$$

$$R_{xx}(\tau) = \begin{cases} \dfrac{e^{-a\tau}}{2a}, & \tau \geq 0 \\ \dfrac{e^{a\tau}}{2a}, & \tau < 0 \end{cases}$$

$$\boxed{R_{xx}(\tau) = \frac{1}{2a} e^{-a|\tau|}}$$

Auto-correlation fun. of $e^{-at} u(t)$



Decaying exponential in +ve, -ve axis.

ESD :

$$x(t) = e^{-at} u(t)$$

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j2\pi F t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j2\pi F)t} dt$$

$$= \left. \frac{e^{-(a+j2\pi F)t}}{-(a+j2\pi F)} \right|_{0}^{\infty}$$

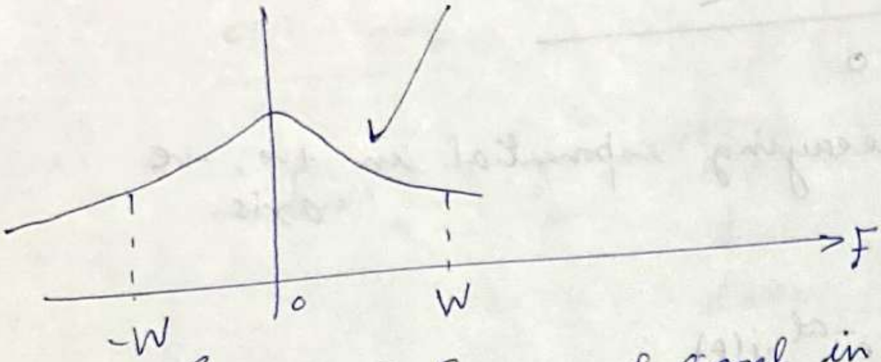
$$X(F) = \frac{1}{a+j2\pi F}$$

$$S_{xx}(F) = |X(F)|^2 = \left| \frac{1}{a+j2\pi F} \right|^2$$

$$R_{xx}(\tau) \xleftrightarrow{F.T.} \boxed{S_{xx}(F) = \frac{1}{a^2 + 4\pi^2 F^2}}$$

ESD.

$$S_{xx}(F) = \frac{1}{a^2 + 4\pi^2 F^2}$$



Spread of Energy of signal in Frequency Domain.

Energy contained in the band $[-W, W]$

$$\int_{-W}^{W} \frac{dF}{a^2 + 4\pi^2 F^2}$$

$2\pi F = \tilde{F} \Rightarrow 2\pi\, dF = d\tilde{F}$

$$= \int_{-W}^{W} \frac{dF}{a^2 + (2\pi F)^2} = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} \frac{d\tilde{F}}{\left[a^2 + (\tilde{F})^2\right]}$$

$$= \frac{1}{2\pi a} \tan^{-1}\left(\frac{\tilde{F}}{a}\right) \Bigg|_{-2\pi W}^{2\pi W}$$

$$= \frac{1 \times 2}{2\pi a} \left[\tan^{-1}\left(\frac{2\pi W}{a}\right)\right]$$

$$= \frac{1}{\pi a} \left[\tan^{-1}\left(\frac{2\pi W}{a}\right)\right]$$

Energy of $x(t)$ contained in the band $[-W, W]$ is :-

$$\frac{1}{\pi a} \tan^{-1}\left[\frac{2\pi W}{a}\right]$$

Say $W \to \infty$

$$\frac{1}{\pi a} \tan^{-1}\left[\frac{2\pi W}{a}\right]\Bigg|_{W \to \infty} = \frac{1}{\pi a} \times \frac{\pi}{2}\Bigg|_{W \to \infty} = \frac{1}{2a}$$

Total Energy

$$x(t) = e^{-at} u(t)$$

$$E_x = \int_0^\infty e^{-2at}\, dt = \frac{e^{-2at}}{-2a}\Bigg|_0^\infty = \frac{1}{2a}$$

$R_{xx}(\tau) \leftarrow$ Set $\tau = 0$

$$\underbrace{\int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt}_{\text{Set } \tau=0}$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$$

$$\Rightarrow \boxed{R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} S_{xx}(F)\, dF = \int_{-\infty}^{\infty} |X(F)|^2\, dF}$$

$$= \cancel{\int_{-\infty}^{\infty} S_{xx}(F)\, dF}$$

(Parseval's Relation / Theorem)

Auto-correlation

$x_1(t)$

$$R_{11}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_1^*(t-\tau)\, dt$$

Or

$x(t)$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt \;=\; x(t) * x^*(-t)$$

$$S_{xx}(F) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j2\pi F\tau}\, d\tau \;=\; X(F) \cdot FT\left[x^*(-\tau)\right]$$

$$= X(F) \cdot X^*(F) = |X(F)|^2$$

$$\boxed{S_{xx}(F) = |X(F)|^2}$$

Fourier Transform of auto-correlation.

$X(F)$: $FT\left[x(t)\right]$

$$R_{xx}(\tau) \longleftrightarrow S_{xx}(F)$$

$R_{xx}(\tau)$ is IFT of $S_{xx}(F)$

$$\Rightarrow R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(F)\, e^{j2\pi F\tau}\, dF$$

$$\Rightarrow \boxed{R_{xx}(0) = \int_{-\infty}^{\infty} S_{xx}(F)\, dF = \int_{-\infty}^{\infty} |X(F)|^2\, dF}$$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t-\tau)\, dt$$

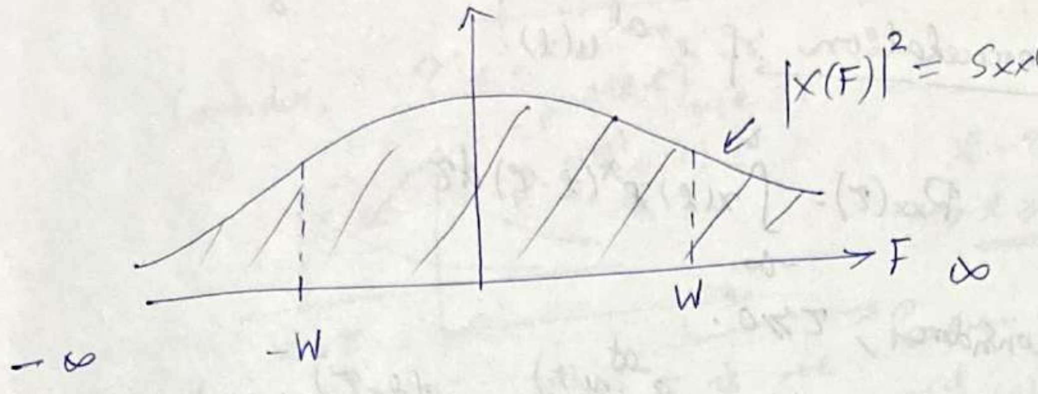
$$R_{xx}(0) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$\therefore \boxed{R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} S_{xx}(F)\, dF = \int_{-\infty}^{\infty} |X(F)|^2\, dF}$$

Energy of $x(t)$

$$\boxed{\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF}$$

Parseval's relation for a continuous time signal



Total Energy $= \int_{-\infty}^{\infty} |X(F)|^2 dF = \int_{-\infty}^{\infty} S_{xx}(F)\, dF$

Total Energy of the signal contained in the band $[-W, W]$

$$\int_{-W}^{W} S_{xx}(F)\, dF =$$

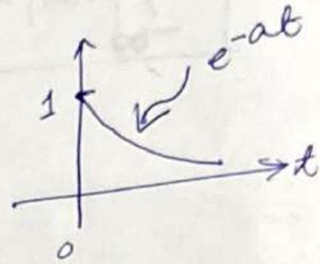
Energy Distribution of $x(t)$ in Frequency Domain.

$S_{xx}(F) = |X(F)|^2$: Energy Spectral Density (ESD)

$\boxed{S_{xx}(F) \geq 0 \text{ for all } F}$ → Non-negative.

Example of Auto-correlation

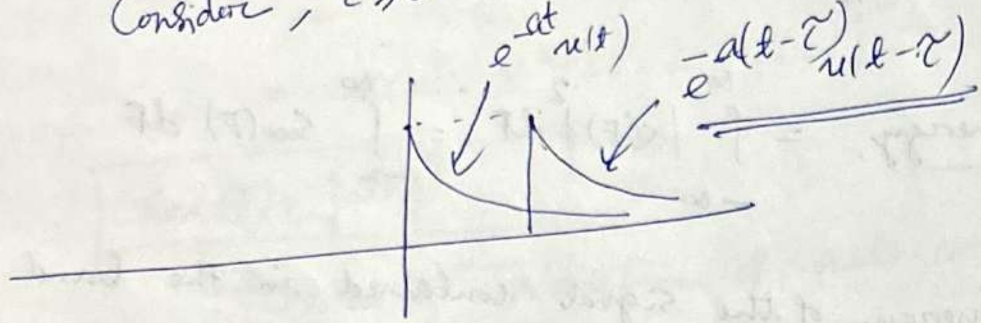
$$x(t) = e^{-at}u(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$



Auto-correlation of $e^{-at}u(t)$

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, d\tau$$

Consider, $\tau \geq 0$.



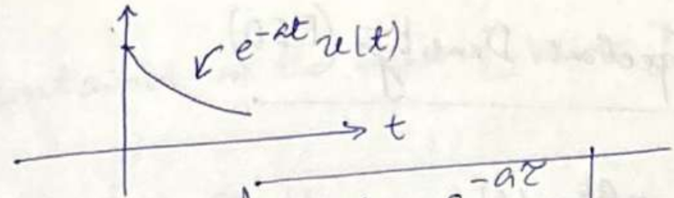
$$R_{xx}(\tau) = \int_{-\infty}^{\infty} e^{-at}u(t) \cdot \underbrace{e^{-a(t-\tau)} u(t-\tau)}_{\text{non-zero only for } t \geq \tau}\, d\tau$$

$$= \int_{\tau}^{\infty} e^{-at} \cdot e^{-a(t-\tau)}\, d\tau$$

$$= e^{+a\tau} \int_{\tau}^{\infty} e^{-2at}\, dt = e^{-a\tau} \left. \frac{e^{-2at}}{-2a} \right|_{\tau}^{\infty}$$

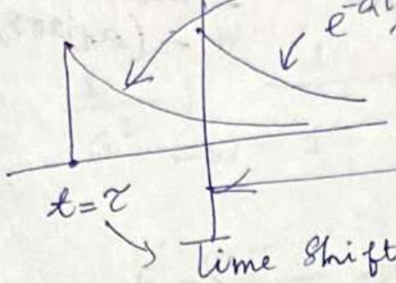
$$= \frac{1}{2a} e^{-a\tau}$$

$$\boxed{R_{xx}(\tau) = \frac{e^{-a\tau}}{2a}} \quad \text{iff } \tau \geq 0$$



If $\tau \geq 0$, $\boxed{R_{11}(\tau) = \frac{e^{-a\tau}}{2a}}$

Consider, $\tau \leq 0$



$t - \tau \geq 0$

$\Rightarrow t \geq \tau$

$t=\tau \rightarrow$ time shift τ is $-ve$

$\rightarrow$ Portion of overlap of $e^{-at}u(t)$ & $e^{-a(t-\tau)}u(t-\tau)$ for $t \geq 0$

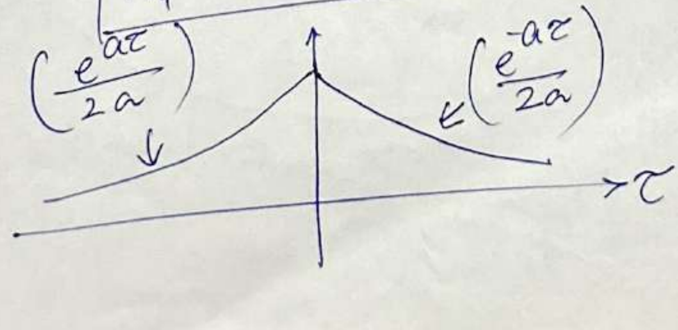
$$R_{11}(\tau) = \int_0^\infty e^{-at} e^{-a(t-\tau)} d\tau$$

$$= e^{a\tau}\int_0^\infty e^{-2at} dt = e^{a\tau} \frac{e^{-2at}}{-2a}\Big|_0^\infty$$

$$\boxed{R_{11}(\tau) = \frac{e^{a\tau}}{2a}}, \text{ for } \tau < 0$$

$$R_{11}(\tau) = \begin{cases} \frac{1}{2a}e^{-a\tau}, & \tau \geq 0 \\ \frac{1}{2a}e^{a\tau}, & \tau < 0 \end{cases} = \frac{1}{2a}\left[e^{-a|\tau|}\right]$$

$$\boxed{R_{11}(\tau) = \frac{1}{2a}e^{-a|\tau|}}$$ $\leftarrow$ ACF of $e^{-at}u(t)$



Energy Spectral Density (ESD)

$$x(t) = e^{-at}u(t)$$

$$X(F) = \int_0^{\infty} e^{-at} e^{-j2\pi Ft}\,dt$$

$$= \int_0^{\infty} e^{-(a+j2\pi F)t}\,dt = \left.\frac{e^{-(a+j2\pi F)t}}{-(a+j2\pi F)}\right|_0^{\infty}$$

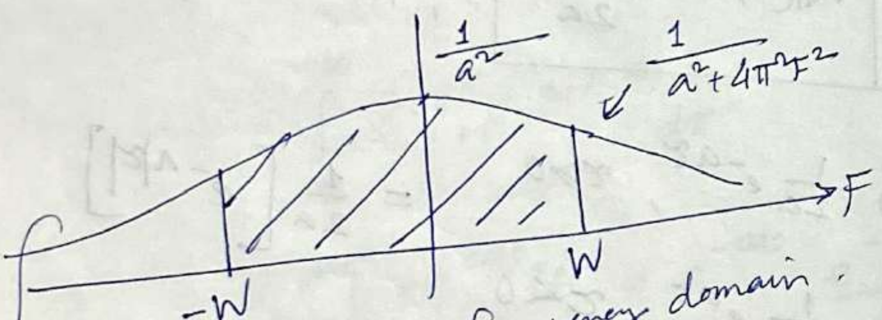
$$\Rightarrow \boxed{X(F) = \frac{1}{a+j2\pi F}}$$

ESD, $S_{11}(F) = |X(F)|^2 = \left|\frac{1}{a+j2\pi F}\right|^2 = \left(\frac{1}{\sqrt{a^2+4\pi^2F^2}}\right)^2$

$$\Rightarrow \boxed{S_{11}(F) = \frac{1}{a^2+4\pi^2F^2}}$$

$$\big\updownarrow \text{F.T.}$$

$$R_{xx}(\tau)$$



Spread of energy in frequency domain.

Energy contained in the band $[-W, W]$

$$\int_{-W}^{W} S_{xx}(F)\, dF$$

$$= \int_{-W}^{W} \frac{1}{a^2 + 4\pi^2 F^2}\, dF$$

$$= \frac{1}{a^2} \int_{-W}^{W} \frac{1}{1 + \frac{F^2}{a^2/4\pi^2}}\, dF$$

$$\tilde{F} = \frac{F}{a/2\pi} \Rightarrow d\tilde{F} = \frac{2\pi}{a}\, dF$$

$$= \frac{1}{a^2} \cdot \frac{a}{2\pi} \int_{-\frac{2\pi W}{a}}^{\frac{2\pi W}{a}} \frac{1}{1 + (\tilde{F})^2}\, d\tilde{F}$$

$$= \frac{1}{2\pi a} \int_{-\frac{2\pi W}{a}}^{\frac{2\pi W}{a}} \frac{d\tilde{F}}{1 + \tilde{F}^2} = \frac{1}{2\pi a} \tan^{-1}(\tilde{F}) \Big|_{-\frac{2\pi W}{a}}^{\frac{2\pi W}{a}}$$

$$= \frac{1}{2\pi a} \times 2 \tan^{-1}\left(\frac{2\pi W}{a}\right)$$

$$= \frac{1}{\pi a} \tan^{-1}\left(\frac{2\pi W}{a}\right)$$

→ Energy of $x(t) = e^{-at} u(t)$ in the band $[-W, W]$.

Set $W \to \infty$

$$E_x = \frac{1}{\pi a} \tan^{-1}\left(\frac{2\pi W}{a}\right)\Bigg|_{W \to \infty}$$

$$= \frac{1}{\pi a} \times \frac{\pi}{2}$$

$$= \frac{1}{2a}$$

$x(t) = e^{-at} u(t)$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{0}^{\infty} e^{-2at} dt$$

$$= \frac{e^{-2at}}{-2a}\Bigg|_0^{\infty}$$

$$= \frac{1}{2a}.$$

ACF

$$R_{xx}(\tau) \xrightarrow{\text{Set } \tau = 0} \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Set $\tau = 0$

$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$$

$$\therefore R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Total Energy of signal

$x(t) = e^{-at} u(t)$

$R_{xx}(\tau) = \frac{1}{2a} e^{-a|\tau|}$

$R_{xx}(0) = \frac{1}{2a}$

$$R_{xx}(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} S_{xx}(F)\, dF = \int_{-\infty}^{\infty} |X(F)|^2 dF$$

Parseval's Relation.

AMPLITUDE MODULATION (AM)

$$c(t) = A_c \cos(2\pi F_c t)$$

$c(t) \rightarrow$ Carrier

$A_c \rightarrow$ Amplitude of Carrier

$F_c \rightarrow$ Carrier frequency

AM : Message modulates the amplitude of carrier.

$$x(t) = (1 + K_a m(t))\, A_c \cos(2\pi F_c t)$$

$$\Rightarrow \boxed{x(t) = A_c [1 + K_a m(t)] \cos(2\pi F_c t)}$$

$\downarrow$

Amplitude Modulated Signal

$K_a \rightarrow$ Sensitivity ~~of~~

$m(t) \rightarrow$ Message Signal

Amplitude of Carrier is modulated according to message $m(t)$.

Example :-

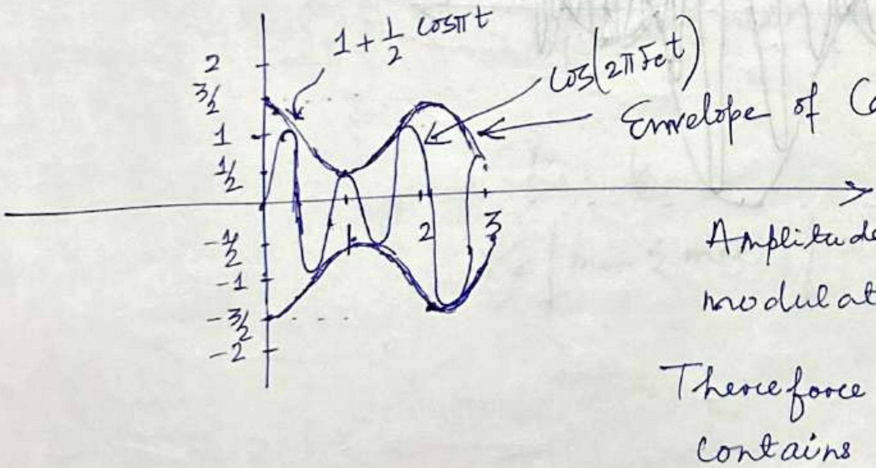
$A_c = 1$

$K_a = \frac{1}{2}$ $\quad F_m = \frac{1}{2}$

$$m(t) = \cos(2\pi F_m t)$$
$$= \cos(\pi t)$$

AM : $x(t) = A_c(1 + K_a m(t)) \cos(2\pi F_c t)$

$$\Rightarrow x(t) = \left(1 + \frac{1}{2}\cos \pi t\right) \cos(2\pi F_c t)$$



Envelope of Carrier

Amplitude of the carrier is modulated by message.

Therefore envelope of carrier contains information about message.

$$\boxed{F_c \gg F_m}$$

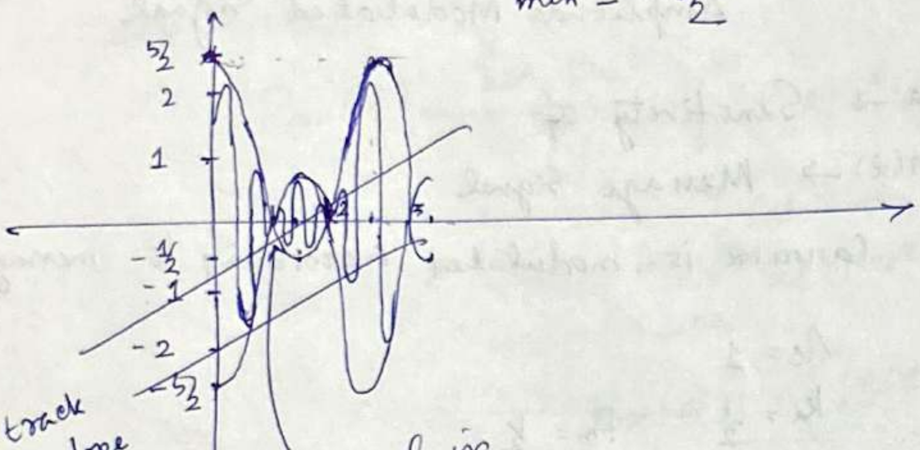
Carrier Frequency — maximum freq. Component of the message.

$\underline{K_a = 3/2}$

$1 + K_a\, m(t) = 1 + \underbrace{\tfrac{3}{2}\cos(\pi t)}$

max = 5/2

min = −1/2



As we track +ve envelope

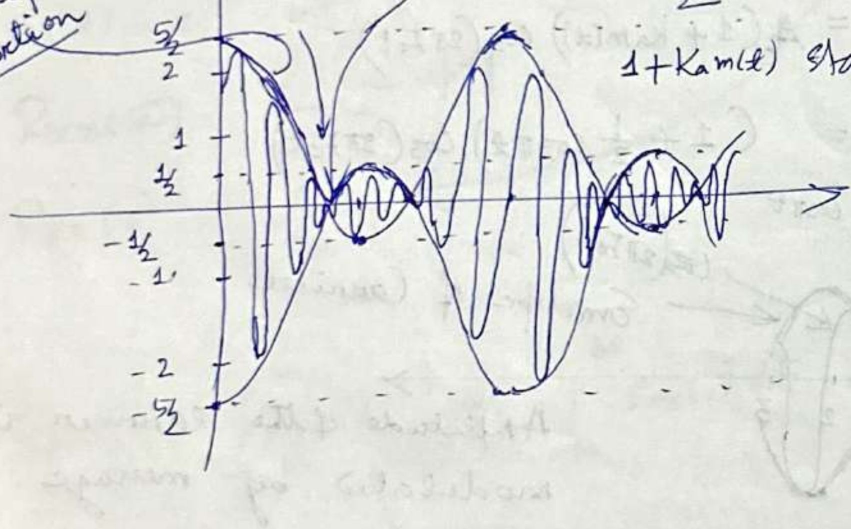
envelope distortion

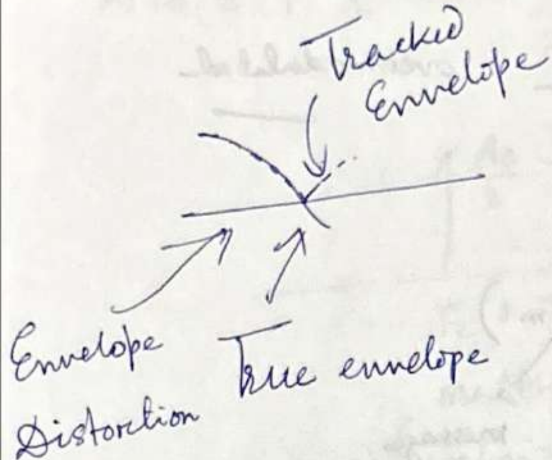
Phase reversal in modulating signal

$1 + K_a\, m(t)$

$= 1 + \tfrac{3}{2}\cos(\pi t) = 0$

$1 + K_a m(t)$ starts becoming −ve





To avoid envelope distortion

Chose K_a appropriately

$$1 + K_a\, m(t) \geq 0$$

For No distortion

$$\Rightarrow K_a m(t) \geq -1$$

$$\Rightarrow \boxed{K_a \left| \min\{m(t)\} \right| \leq 1}$$

$$\mu \leq 1$$

Modulation Index

$$\mu = K_a |m(t)|_{min}$$

$$= K_a \left| \min\{m(t)\} \right|$$

For no envelope distortion $\mu \leq 1$

If $\mu > 1 \rightarrow$ Carrier is overmodulated.

$\Downarrow$

Envelope Distortion.

$$m(t) = \cos \pi t$$

$$\left| \min\{m(t)\} \right| = \left| \min\{\cos \pi t\} \right| = |-1| = 1$$

$$K_a = 1/2$$

$$K_a \left| \min\{m(t)\} \right| = \frac{1}{2} \times 1 = 1/2 < 1$$

$K_a = 3/2$

$K_a \left| \min\{m(t)\} \right| = \frac{3}{2} \times 1 = 3/2 > 1$: overmodulated

For a sinusoidal message signal

$m(t) = A_m \cos(2\pi F_m t)$

A_m ↓ Amplitude of message

F_m ↘ message frequency

$\left| \min\{m(t)\} \right| = \left| -A_m \right| = A_m$

$\boxed{\mu = K_a A_m}$

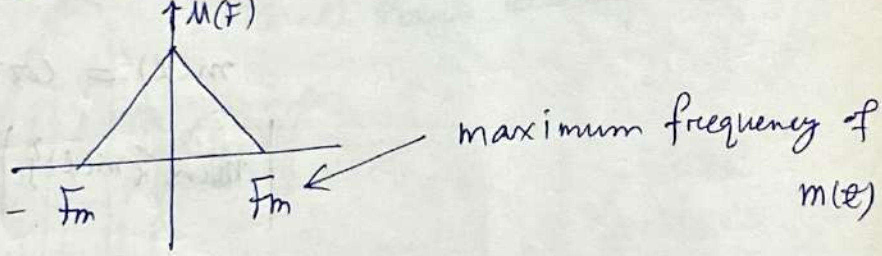
Modulation Index = Sensitivity Factor × Amplitude of message

For a sinusoidal message, $m(t) = A_m \cos(2\pi F_m t)$

Spectrum of an AM signal :

message : $m(t)$

& $m(t) \longleftrightarrow M(F)$



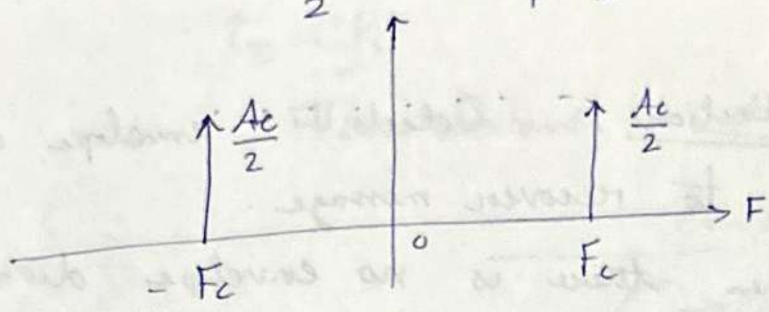
$x(t) = A_c \left(1 + K_a m(t)\right) \cos(2\pi F_c t)$

$= A_c \cos(2\pi F_c t) + K_a A_c\, m(t) \cdot \cos(2\pi F_c t)$

$\cos 2\pi F_c t = \dfrac{e^{j2\pi F_c t} + e^{-j2\pi F_c t}}{2} \longleftrightarrow \frac{1}{2}\left[\delta(F - F_c) + \delta(F + F_c)\right]$

$e^{j2\pi F_c t} \longleftrightarrow \delta(F - F_c)$, $e^{-j2\pi F_c t} \longleftrightarrow \delta(F + F_c)$

$$A_c \cos(2\pi F_c t) \longleftrightarrow \frac{A_c}{2}\delta(F-F_c) + \frac{A_c}{2}\delta(F+F_c)$$



multiplication in time domain

$$\underbrace{A_c K_a m(t)}\cdot\underbrace{\cos(2\pi F_c t)}$$

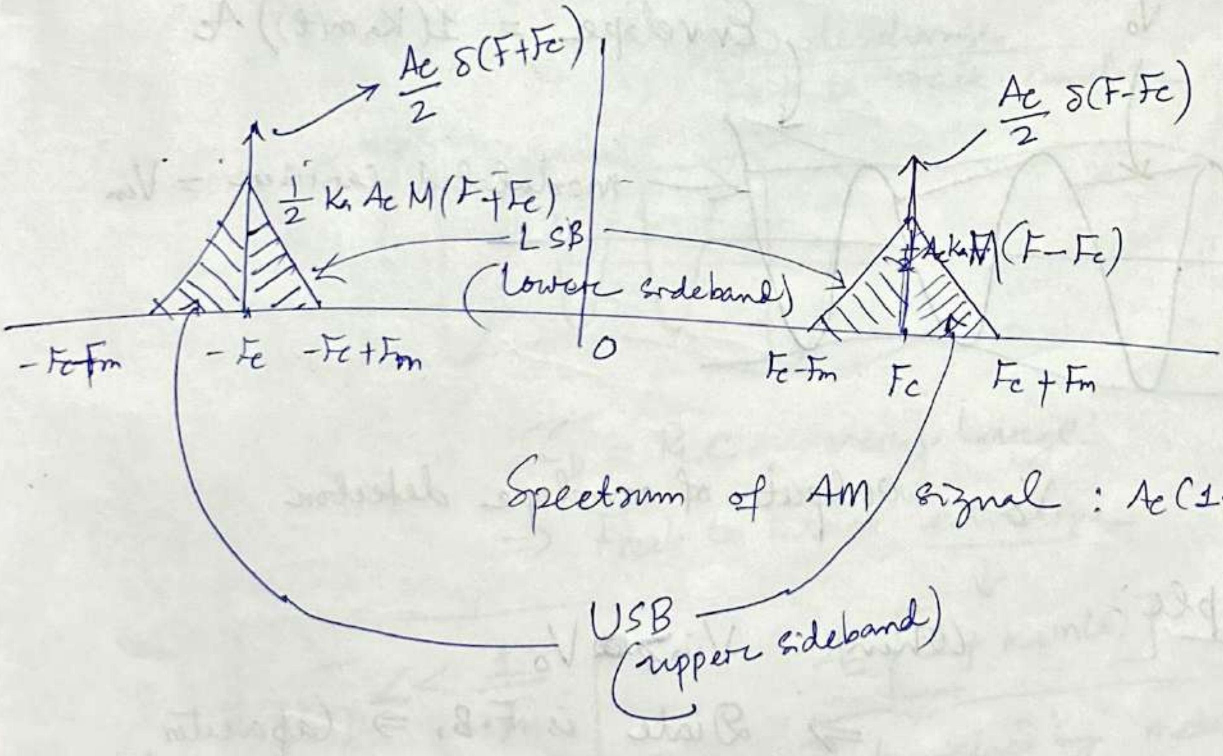
$$\updownarrow \qquad\qquad \updownarrow$$

$$A_c K_a M(F) * \frac{1}{2}\left[\delta(F-F_c)+\delta(F+F_c)\right]$$

Convolution in Frequency Domain.

$$= \frac{A_c K_a}{2} M(F) * \delta(F-F_c) + \frac{A_c K_a}{2} M(F) * \delta(F+F_c)$$

$$= \frac{A_c K_a}{2} M(F-F_c) + \frac{A_c K_a}{2} M(F+F_c)$$

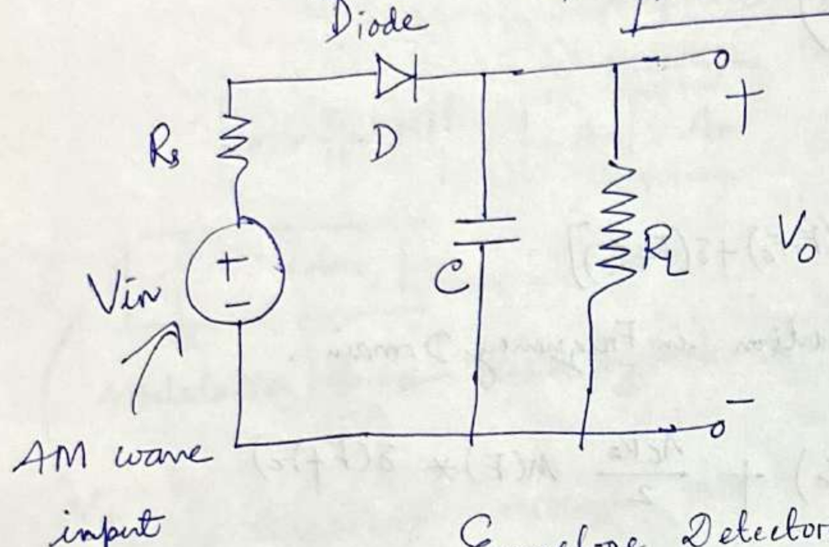


Spectrum of AM signal : $A_c(1+K_a m(t))\cos(2\pi F_c t)$

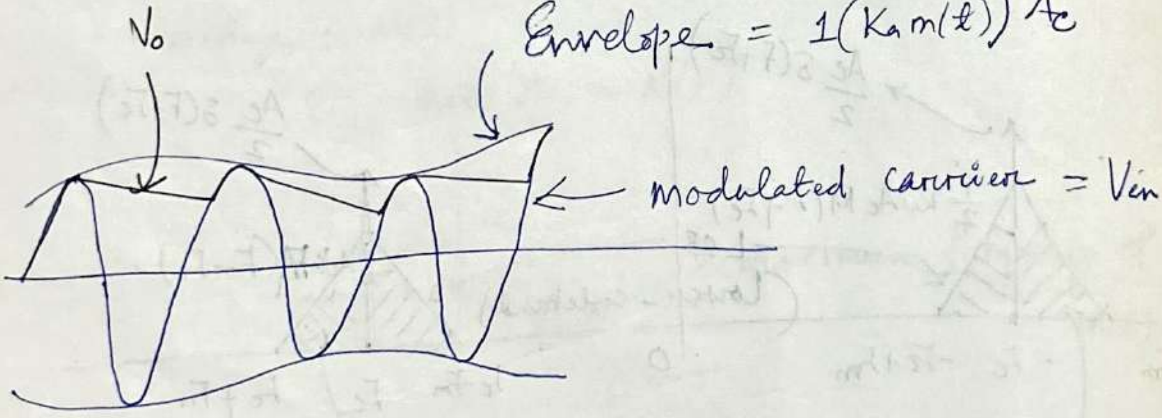
Detection of AM

Envelope Detection:
Detects the envelope of transmitted signal to recover message.

→ Works only when there is no envelope distortion $\Rightarrow \boxed{\mu \leq 1}$



Envelope Detector or
Peak Detector



V_o = Output of envelope detector

Principle:

When $V_{in} > V_o$

$\Rightarrow$ Diode is F.B. $\Rightarrow$ Capacitor charges to peak of V_{in}

$V_{in} < V_o$

$\Rightarrow$ D is R.B. $\Rightarrow$ Capacitor discharges through R_L

Capacitor discharges slowly

$$\underset{\sim}{\tau_D} = CR_L$$

Time constant for discharging.

Envelope Detector

Low cost receiver
- → Less complex
- → Low cost
- → Easy to implement
- → Long distance capability

(MF) 300KHz - 3MHz | (HF) (3-30MHz)

Broadcast application. | over a long distance

Ex:- Radio, TV

→ Between Aircraft & air traffic control.

→ Maritime communication
Ship-to-ship & ship-to-shore
(MF) & (HF) bands

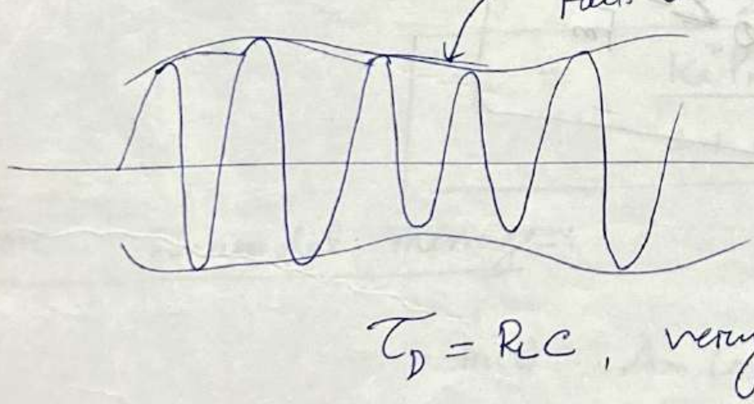
→ Military Radio

$$V_{in} = A_c(1 + K_a m(t)) \cos(2\pi F_c t)$$

$$\tau_D = R_L C$$

$V_{in} < V_0$: Diode is R.B.

Slow discharge
Fails to track envelope



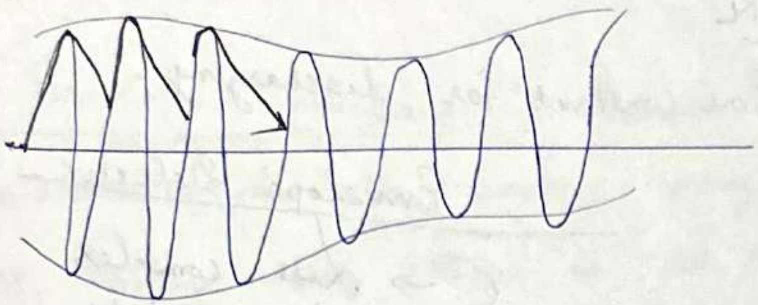
$\tau_D = R_L C$, very Large

$\Rightarrow$ Fails to track envelope

$$\underbrace{A_c\left[1 + K_a m(t)\right]}_{\text{Determined by message } m(t)}$$

$$\boxed{\tau_D << \frac{1}{F_m}}$$

Discharge is Faster than the rate of change of envelope



If τ_D is too small
$\Rightarrow$ capacitor discharges very rapidly.
Output of the capacitor tracks the carrier & not the envelope.

$$\boxed{\tau_D = R_L C \gg \frac{1}{F_c}}$$

so that o/p ~~doesnot~~ tracks envelope rather than carrier

$$\boxed{\frac{1}{F_c} \ll \tau_D \ll \frac{1}{F_m}}$$

Power of an AM Signal

$$x(t) = A_c[1 + K_a m(t)] \cos(2\pi F_c t)$$

$$= \underbrace{A_c \cos(2\pi F_c t)}_{\downarrow} + \underbrace{A_c K_a m(t) \cos 2\pi f_c t}_{\downarrow}$$

$$\text{Avg Power} = \frac{A_c^2}{2}$$

$$\text{Inst. power} : \frac{K_a^2 A_c^2 m^2(t)}{2}$$

$$\text{Avg Power} : E\left\langle \frac{1}{2} K_a^2 A_c^2 m^2(t) \right\rangle$$

$$= \frac{1}{2} K_a^2 A_c^2 E\langle m^2(t) \rangle$$

$$= \frac{1}{2} K_a^2 A_c^2 P_m$$

{ $P_m \rightarrow$ Avg power of message signal }

$$P_T = \frac{A_c^2}{2} + \frac{K_a^2 A_c^2 P_m}{2}$$

$$\eta = \text{efficiency of AM} = \frac{\text{Power in carrier modulated by message}}{\text{Total Power}}$$

$$= \frac{\frac{1}{2} A_c^2 K_a^2 P_m}{\frac{1}{2} A_c^2 + \frac{1}{2} A_c^2 K_a^2 P_m}$$

$$\boxed{\eta = \frac{K_a^2 P_m}{1 + K_a^2 P_m}}$$

For sinusoidal message:

$$m(t) = A_m \cos(2\pi F_m t)$$

$$P_m = \frac{A_m^2}{2}$$

$$\eta = \frac{K_a^2 \frac{A_m^2}{2}}{1 + K_a^2 \frac{A_m^2}{2}} = \frac{(K_a A_m)^2}{2 + (K_a A_m)^2}$$

$$\Rightarrow \boxed{\eta = \frac{\mu^2}{2 + \mu^2}}$$

$$\eta = \frac{\mu^2}{2+\mu^2} = 1 - \underbrace{\frac{2}{2+\mu^2}}_{\text{decreases with } \mu}$$

η increases with μ

For envelope detection, $\max(\mu) = 1$

$$\eta_{max} = \left.\frac{\mu^2}{\mu^2+2}\right|_{\mu=1} = \frac{1}{2+1} = \frac{1}{3}$$

$$\boxed{\% \eta_{max} = 33\%}$$

maximum efficiency with no ED

Or

$$s(t) = A_C\left[1 + \mu \cos 2\pi F_m t\right] \cos(2\pi F_c t)$$

$$= A_c \cos(2\pi F_c t) + \mu A_c \cos(2\pi F_c t) \cos(2\pi F_m t)$$

$$= A_c \cos(2\pi F_c t) + \frac{\mu A_c}{2} \cos 2\pi (F_c + F_m)t + \frac{\mu A_c}{2} \cos(2\pi \{F_c - F_m) t)$$

$$P_C = \frac{A_c^2}{2} \qquad P_{USB} = \frac{\frac{\mu^2 A_c^2}{2^2}}{2} = \frac{\mu^2 A_c^2}{8} \qquad P_{LSB} = \frac{\frac{\mu^2 A_c^2}{2^2}}{2} = \frac{\mu^2 A_c^2}{8}$$

$$\eta = \frac{P_{LSB} + P_{USB}}{P_C + P_{LSB} + P_{USB}} = \frac{\frac{\mu^2 A_c^2}{8} + \frac{\mu^2 A_c^2}{8}}{\frac{A_c^2}{2} + \frac{\mu^2 A_c^2}{8} + \frac{\mu^2 A_c^2}{8}}$$

$$\Rightarrow \eta = \frac{\frac{\mu^2 A_c^2}{4}}{\frac{A_c^2}{2} + \frac{\mu^2 A_c^2}{4}} = \frac{\mu^2}{2+\mu^2}$$

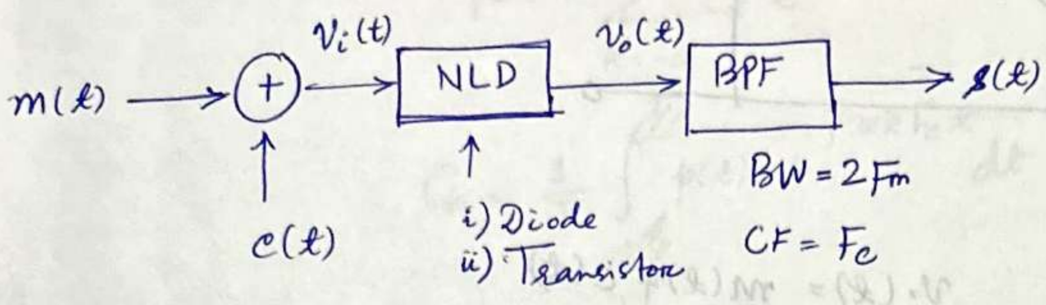
AM Modulation

2 methods:

a) Square Law Modulation

b) Switching Modulation.

a) Square Law Modulation



$$v_i(t) = m(t) + c(t) = m(t) + A_c' \cos(2\pi F_c t)$$

$$v_o(t) = a_1 v_i(t) + a_2 v_i^2(t)$$

$$= a_1 \left[m(t) + A_c' \cos(2\pi F_c t)\right] + a_2 \left[m(t) + A_c' \cos(2\pi F_c t)\right]^2$$

$$\Rightarrow v_o(t) = a_1 m(t) + a_1 A_c' \cos(2\pi F_c t) + a_2 m^2(t) + a_2 A_c'^2 \cos^2(2\pi F_c t) + 2 a_2 m(t) \cdot A_c' \cos(2\pi F_c t)$$

O/P of BPF:

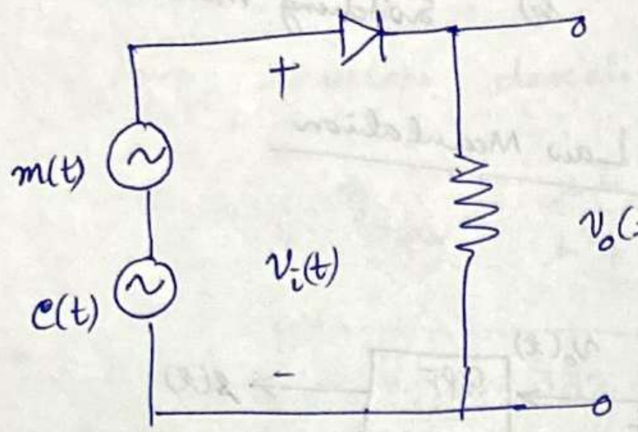
$$s(t) = a_1 A_c' \cos(2\pi F_c t) + 2a_2 m(t) \cdot A_c' \cos(2\pi F_c t)$$

$$= a_1 A_c' \left[1 + \frac{2a_2 A_c'}{a_1 A_c'} m(t)\right] \cos(2\pi F_c t)$$

$$= a_1 A_c' \left[1 + \frac{2a_2}{a_1} m(t)\right] \cos(2\pi F_c t)$$

$$\therefore s(t) = A_c\left[1 + K_a m(t)\right] \cos(2\pi F_c t) \qquad \begin{cases} A_c = a_1 A_c' \\ K_a = \dfrac{2a_2}{a_1} \end{cases}$$

b) Switching Modulator



$$v_i(t) = m(t) + c(t)$$

$$= m(t) + A_c \cos(2\pi F_c t)$$

Since $A_c \gg$ amplitude $\{m(t)\}$

Carrier $c(t)$ decides the states of diode. (ON or OFF)

So, $v_i(t) \geq 0 \implies v_o(t) = v_i(t)$

& if $v_i(t) < 0 \implies v_o(t) = 0$

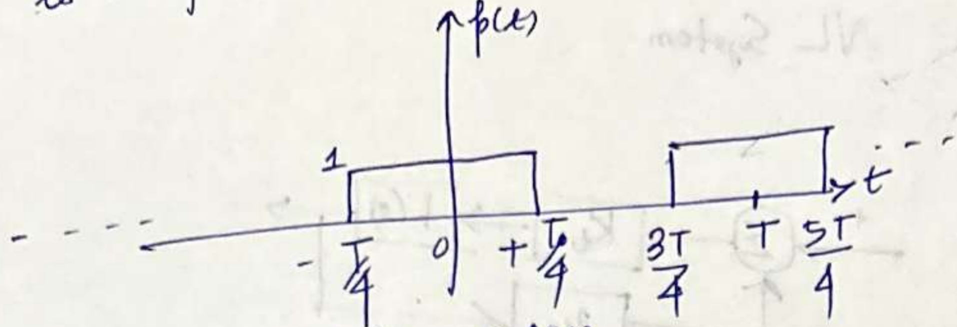
So $v_o(t)$ varies between $v_i(t)$ & 0 at a rate equal to carrier frequency F_c.

So, $v_o(t) = v_i(t) \cdot p(t)$

$$= \left[m(t) + A_c \cos(2\pi F_c t) \right] p(t)$$

$p(t)$ is a pulse train with duty cycle = 50%.

$\left\{ T = \frac{1}{F_c} \right.$



F.S. representation of $p(t)$

$$p(t) = \sum_{k=-\infty}^{\infty} C_k \, e^{j2\pi k F_c t}$$

$$C_k = \frac{1}{T} \int_{T}^{} p(t) \cdot e^{-j2\pi k F_c t} \, dt$$

$$C_0 = \frac{1}{T} \int_{-T/4}^{T/4} 1 \, dt$$

$$= \frac{1}{T} \times \frac{T}{2}$$

$$= \frac{1}{2}$$

$T \cancel{\operatorname{sinc}(FT)}$

$\cancel{\frac{T}{2} \operatorname{sinc} kF}$

$$= \frac{1}{T} \int_{-T/4}^{T/4} e^{-j2\pi k F_c t} \, dt$$

$$= \frac{1}{T} \left. \frac{e^{-j2\pi k F_c t}}{-j2\pi k F_c} \right|_{(-T/4)}^{(T/4)}$$

$$= \frac{1}{T} \times \frac{1}{j2\pi k F_c} \left\{ e^{j\frac{\pi k F_c T}{2}} - e^{-j\frac{\pi k F_c T}{2}} \right\}$$

$$= \frac{1}{j2\pi k} \left\{ e^{j\frac{\pi k}{2}} - e^{-j\frac{\pi k}{2}} \right\}$$

$$= \frac{1 \times \sin\left(\frac{\pi k}{2}\right)}{\pi k}$$

$p(t) = \frac{1}{2} + \frac{1}{\pi}$

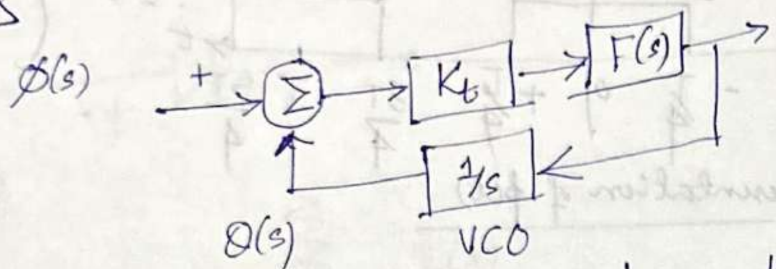
~~$p(t) =$~~

$$C_k = \begin{cases} \frac{1}{2}, & k = 0 \\ 0, & k \to \text{even} \\ \neq 0, & k \to \text{odd} \end{cases}$$

Phase Locked loop

Linear Model

NL System.



$$H(s) = \frac{\theta(s)}{\phi(s)}$$

$$\theta(s) = \frac{K_t F(s)}{s}\left[\phi(s) - \theta(s)\right]$$

$$\Rightarrow \frac{\theta(s)}{\phi(s)} = \frac{\frac{K_t F(s)}{s}}{1 + \frac{K_t F(s)}{s}}$$

$$= \frac{K_t F(s)}{1 + K_t F(s)}$$

Double Sideband Modulation (DSB)

$$x(t) = \underbrace{A_c m(t) \cos(2\pi F_c t)}$$

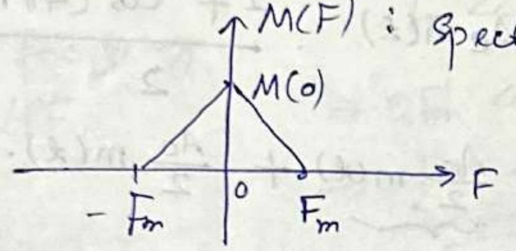
No pure carrier component is present

$m(t) \rightarrow$ Message Signal

$F_c \rightarrow$ Carrier Frequency

$\underbrace{DSB}-\underset{\downarrow}{SC}$

Double Sideband - Suppressed Carrier

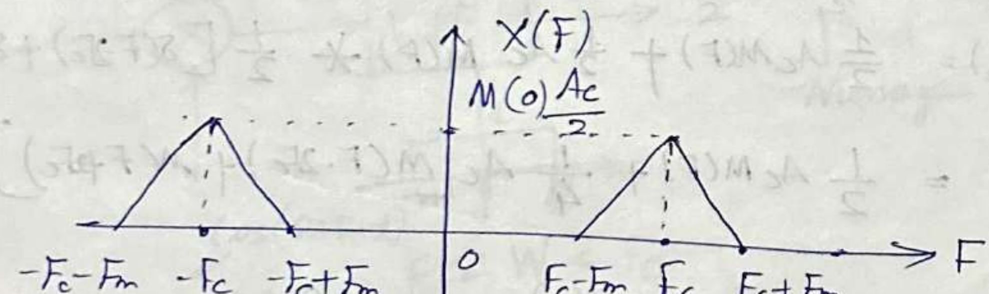


$M(F)$: Spectrum of Message Signal

$$x(t) = A_c m(t) \cos(2\pi F_c t) = m(t) \cdot A_c \cos(2\pi F_c t)$$

$$\Rightarrow X(F) = M(F) * \frac{A_c}{2}\left[\delta(F-F_c) + \delta(F+F_c)\right]$$

$$\Rightarrow X(F) = \frac{A_c}{2}\left[M(F-F_c) + M(F+F_c)\right]$$



$$\boxed{BW = 2F_m}$$

Demodulation of DSB-SC

Demodulation can be carried out by multiplying with a coherently generated carrier $\cos(2\pi F_c t)$ at the receiver.

$$r(t) = x(t) \cdot \cos(2\pi F_c t)$$

$$= A_c\, m(t) \cdot \cos(2\pi F_c t) \cdot \cos(2\pi F_c t)$$

$$= A_c\, m(t) \cdot \cos^2(2\pi F_c t)$$

$$= A_c\, m(t) \cdot \frac{1 + \cos(4\pi F_c t)}{2}$$

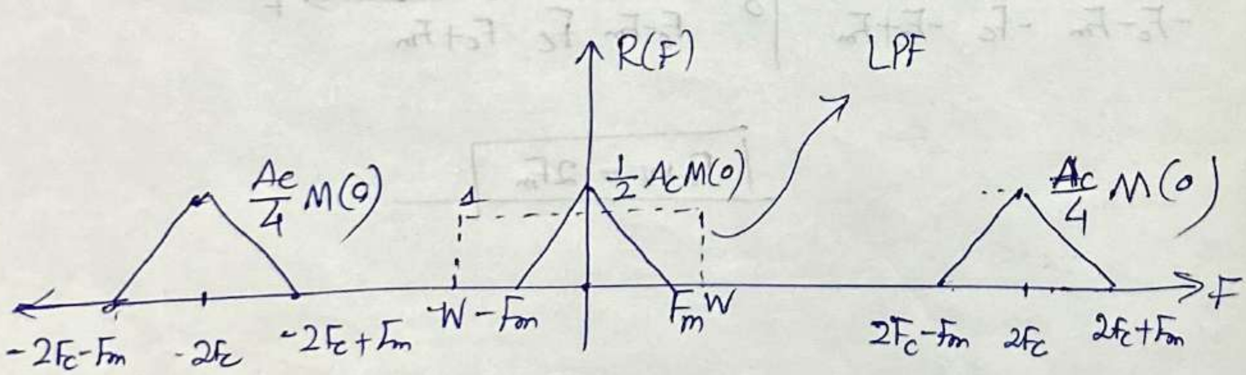
$$= \underbrace{\frac{A_c}{2} m(t)}_{\text{Baseband}} + \underbrace{\frac{A_c}{2} m(t) \cdot \cos(4\pi F_c t)}_{\text{Centred at } \pm 2F_c}$$

Freq $= 2F_c$

$$r(t) = x(t)\cos(2\pi F_c t) = \frac{1}{2} A_c m(t) + \frac{1}{2} A_c m(t) \cos(4\pi F_c t)$$

$$R(F) = \frac{1}{2} A_c M(F) + \frac{1}{2} A_c M(F) * \frac{1}{2}\left[\delta(F-2F_c) + \delta(F+2F_c)\right]$$

$$= \frac{1}{2} A_c M(F) + \frac{1}{4} A_c \left[M(F-2F_c) + M(F+2F_c)\right]$$



Choosing an LPF of suitable bandwidth W

$\underbrace{\frac{A_c}{2} m(t)}_{\text{Base-band}}$ can be seperated form $\underbrace{\frac{A_c}{2} \cos(4\pi F_c t)}_{\text{Sidebands at } \pm 2F_c}$.

Chose LPF, such that cut off frequency,

$$\underline{F_m \le W \le 2F_c - F_m}$$

This is possible because, we have

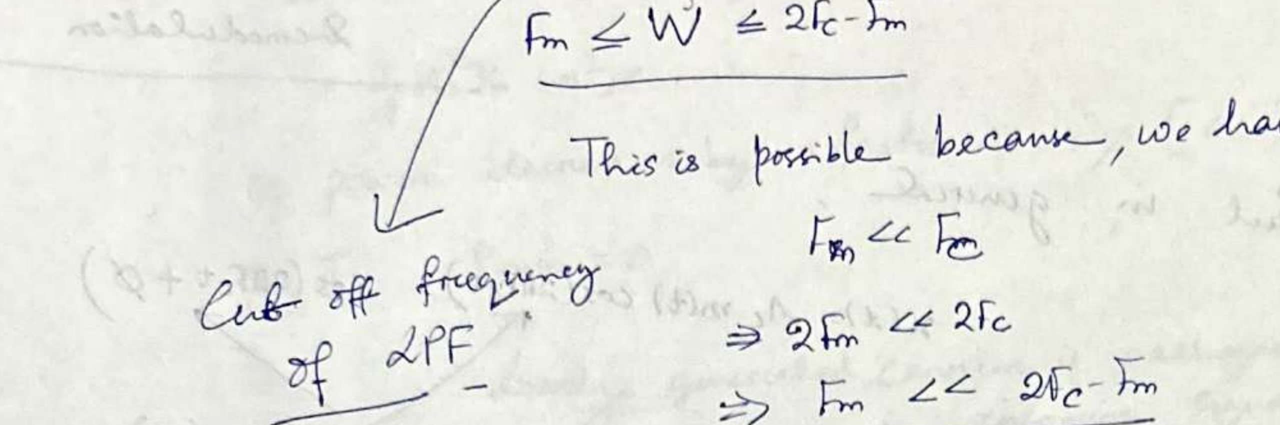
$$F_m << F_c$$

$$\Rightarrow 2F_m << 2F_c$$

$$\Rightarrow \underline{F_m << 2F_c - F_m}$$

Cut off frequency of LPF

LPF blocks the component $\frac{A_c}{2} m(t) \cos(4\pi F_c t)$.



locally generated carrier at Receiver

Schematic Diagram of DSB-SC demodulation process)

* Phase of incoming signal & locally generated carrier at receiver are same.

No Phase offset

↓

Coherent Demodulation.

$x(t) = A_c m(t) \cos(2\pi F_c t)$:- Received Signal

$\times \cos(2\pi F_c t)$ ← Locally generated carrier at receiver

No Phase offset between these 2 → Coherent Demodulation

or Synchronous Demodulation

But in general,

$x(t) = A_c m(t) \cos(2\pi F_c t)$ $\quad \cos(2\pi F_c t + \phi)$

Phase offset (ϕ)

$\phi \rightarrow$ Carrier Phase offset

$\phi \neq 0$

⇒ Carrier at Receiver or Local Oscillator is not coherent w.r.t carrier of incoming signal.

$x(t) \times \cos(2\pi F_c t + \phi)$

$= A_c m(t) \cos(2\pi F_c t) \cdot \cos(2\pi F_c t + \phi)$

$= \frac{A_c m(t)}{2} \left[\cos(4\pi F_c t + \phi) + \cos(\phi) \right]$

$= \underbrace{\frac{A_c}{2} m(t) \cdot \cos\phi}_{\text{Baseband}} + \underbrace{\frac{A_c m(t)}{2} \cos(4\pi F_c t + \phi)}_{\pm 2F_c}$

LPF, Cut off $F_m < W < F_c - F_m$

O/p of LPF : $\frac{1}{2} A_c m(t) \cdot \cos\phi$

Additional factor ϕ. Previously absent in coherent Demodulation.)

Power of Demodulated signal,

$= \frac{1}{4} A_c^2 P_m \cos^2\phi$

O/p power decreases by a factor of $\cos^2\phi$ [≤ 1], for

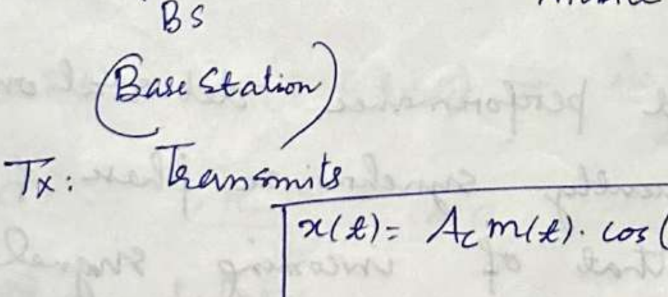
$\phi = \frac{\pi}{2}$, Power $= 0$

Locally generated, carrier is orthogonal to carrier wave in incoming signal.

$\phi\uparrow \Rightarrow P\downarrow \Rightarrow$ Performance of Rx worsens.

It is important to maintain Phase Synchronisation or Phase coherence with respect to carrier wave in incoming signal.

Why does carrier phase offset arises?



Tx: Transmits

$$\boxed{x(t) = A_c m(t) \cdot \cos(2\pi F_c t)}$$

Propagation delay $= \frac{d}{c}$

Signal received at the Mobile user

$$\underset{\downarrow}{y(t)} = \overset{\downarrow \text{ Delayed version of } x(t)}{x(t-\tau)}$$

$$= A_c\, m(t-\tau) \cos[2\pi F_c(t-\tau)]$$

$$= A_c\, m(t-\tau) \cos[2\pi F_c t - 2\pi F_c \tau]$$

$$= A_c\, m(t-\tau) \cos\Big(2\pi F_c t - \underbrace{2\pi F_c \frac{d}{c}}_{\phi \,=\, \text{Phase Offset}}\Big)$$

$m(t-\tau) \simeq m(t)$

(Since, maximum message freq. $F_m << F_m$)

$$\simeq A_c\, m(t) \cos[2\pi F_c t - \phi(d)]$$

$$\boxed{\phi(d) = 2\pi F_c\left(\frac{d}{c}\right)}$$: Phase offset (Varying) depends on distance, d

$\cancel{\frac{\partial \phi(d)}{\partial t} = \frac{\partial}{\partial t}}$

$$\frac{\partial \phi(d)}{\partial t} = \frac{\partial}{\partial t} \frac{2\pi F_c d/c}{dt} = \frac{2\pi F_c}{c} \frac{\partial d}{\partial t}$$

Rate of change of Phase offset $\propto$ velocity of user

Varying distance $\Rightarrow$ varying delay $\Rightarrow$ varying phase offset

Hence, to avoid performance detoriation, one has to frequently synchronize phase of local oscillator with that of incoming signal.

This process is termed as Phase Synchronization.

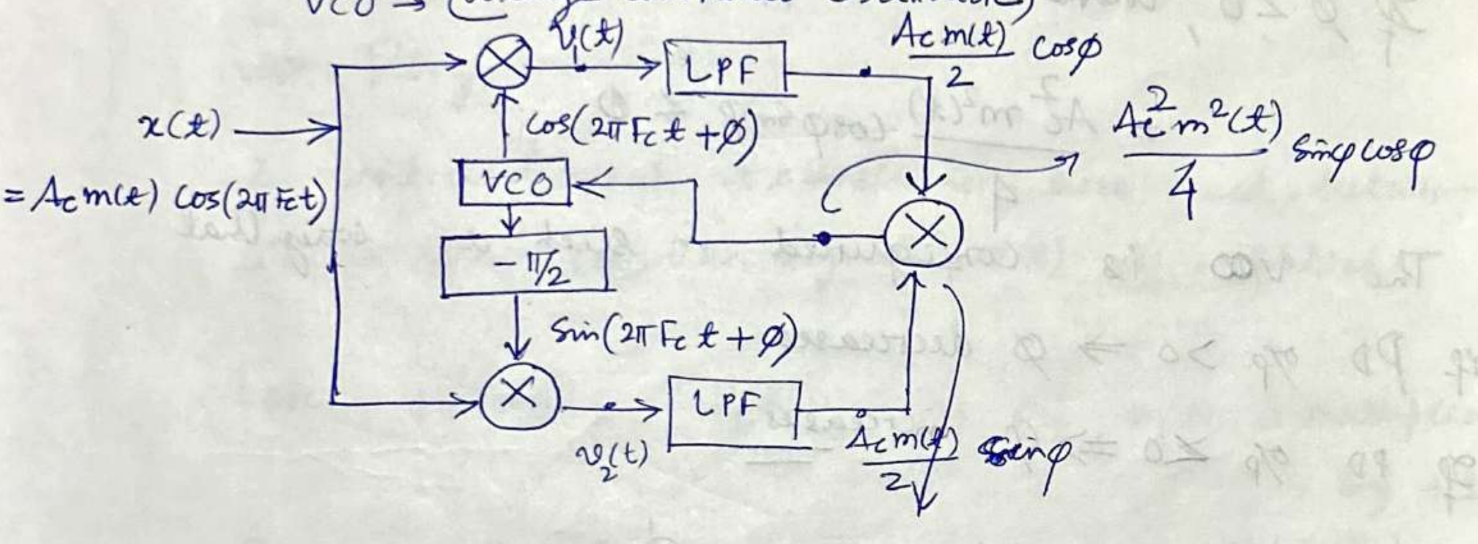
To perform phase synchronization, one can employ Coastas Receiver / Costas Loop.

Costas Receiver enables coherent Demodulation by synchronizing phase of Locally generated carrier with that of the incoming signal.

COSTAS LOOP (OR RECEIVER)

To synchronize Phase of ~~incoming signal to~~ Locally generated carrier with that of incoming signal.

VCO → (Voltage Controlled Oscillator)



Phase Discriminator

upper

$$v_1(t)\ A_c m(t) \cos(2\pi F_c t) \times \cos(2\pi F_c t + \phi) = \frac{A_c m(t)}{2} \left\{ \cos(4\pi F_c t + \phi) + \cos\phi \right\}$$

O/p of LPF : $\frac{A_c m(t)}{2} \cos\phi$

O/p of 2nd demodulator

$$v_2(t) = A_c m(t) \cos(2\pi F_c t) \cdot \sin(2\pi F_c t + \phi)$$

$$= \frac{A_c m(t)}{2}\left[\sin(4\pi F_c t + \phi) + \sin(\phi)\right]$$

Output of LPF :-

$$\frac{A_c m(t)}{2} \sin\phi$$

O/P of Phase Discriminator

$$\frac{A_c^2 m^2(t)}{4} \sin\varphi \cdot \cos\varphi$$

Assume, ϕ to be small.

$$-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$$

If $\phi > 0$, then we have Phase Discriminator (PD) output is

$$\frac{A_c^2 m^2(t)}{4} \cos\varphi \cdot \sin\phi \geq 0$$

If $\phi < 0$, then the PD O/P

$$= \frac{A_c^2 m^2(t)}{4} \cos\varphi \sin\varphi < 0$$

The VCO is configured in such a way that

If PD O/P $> 0 \Rightarrow \phi$ decreases

If PD O/P $< 0 \Rightarrow \varphi$ increases.

Hence +ve ϕ ~~decrea~~ $\Rightarrow$ Decrease in φ

−ve φ $\Rightarrow$ increase in φ

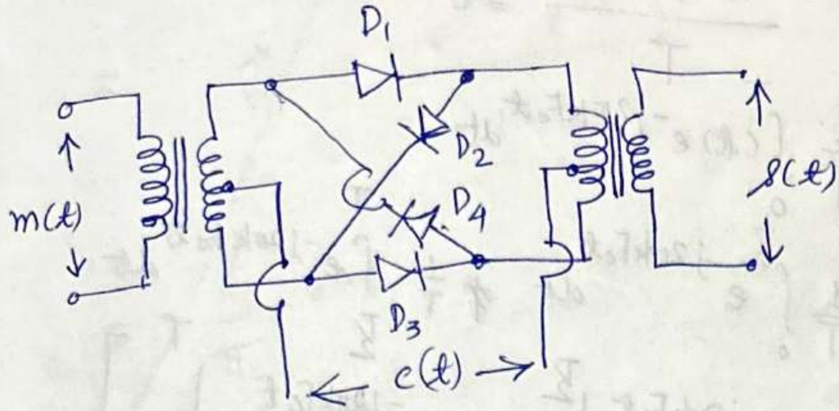
$\Rightarrow$ Phase Offset φ is eventually driven to zero

$\phi = 0$

$\Rightarrow$ ~~Carrier~~ Synchronization is acheived between Locally generated carrier & the incoming signal.

~~Qudrature Carrier Multiplexing~~

RING-MODULATOR



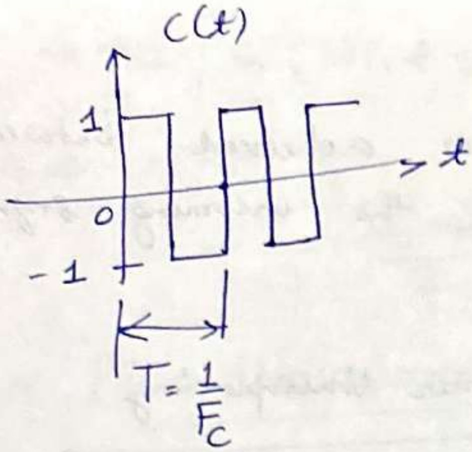
D_1, D_2, D_3, D_4 are connected in a ring structure

$\Rightarrow$ Ring modulator.

2 centre tapped transformers are used, between which the carrier signal $c(t)$ is applied.

For $c(t) > 0$, D_1 & D_3 are ON, $m(t)$ is multiplied by $+1$

For $c(t) < 0$, D_2 & D_4 are ON, $m(t)$ is multiplied by -1



$$C_0 = \frac{(1 \times \frac{T}{2}) + (-1 \times \frac{T}{2})}{T} = 0$$

$$C_k = \frac{1}{T} \int_0^T c(t)\, e^{-j2\pi k F_c t}\, dt$$

$$= \frac{1}{T} \int_0^{T/2} e^{-j2\pi k F_c t}\, dt \; \cancel{+} - \frac{1}{T} \int_{T/2}^{T} e^{-j2\pi k F_c t}\, dt$$

$$= \frac{1}{T} \left[\frac{e^{-j2\pi k F_c t}}{-j2\pi k F_c} \Bigg|_0^{T/2} - \frac{e^{-j2\pi k F_c t}}{-j2\pi k F_c} \Bigg|_{T/2}^{T} \right]$$

$$= \frac{1}{T} \left[\frac{1}{\pi k F_c} \left(\frac{1 - e^{-j\pi k}}{j2} \right) - \frac{1}{\pi k F_c} \left(\frac{e^{-j\pi k} - e^{-j2\pi k}}{j2} \right) \right]$$

$$= \frac{1}{\pi k F_c} \times \frac{1}{T} \left[e^{-j\frac{\pi k}{2}} \sin\left(\frac{\pi k}{2}\right) - e^{-j\frac{3\pi k}{2}} \sin\left(\frac{\pi k}{2}\right) \right]$$

$$= \frac{1}{\pi k} \sin\left(\frac{\pi k}{2}\right) \left[e^{-j\frac{\pi k}{2}} - e^{-j\frac{3\pi k}{2}} \right]$$

$$= \frac{1}{\pi k} \sin\left(\frac{\pi k}{2}\right) e^{-j\pi k} \left(\frac{e^{+j\frac{\pi k}{2}} - e^{-j\frac{\pi k}{2}}}{j2} \right) \times j2$$

$$= \frac{1}{\pi k} \times j2 \times \sin^2\left(\frac{\pi k}{2}\right) e^{-j\pi k} = \begin{cases} 0, & k \to \text{Even} \\ \neq 0, & k \to \text{Odd.} \end{cases}$$

$$= \frac{j2}{\pi k} e^{-j\pi k}$$

$$= \frac{-j2}{\pi k} = \frac{2}{j\pi k}$$

for k (even), $\underline{C_k = 0}$

$C_1 = \frac{1}{\pi} \times j2 \sin^2\left(\frac{\pi}{2}\right) e^{-j\pi}$

$C_{-1} = \frac{1}{-\pi} \times j2 \sin^2\left(\frac{-\pi}{2}\right) e^{j\pi}$

$C_1 + C_{-1} = \cancel{\frac{1}{\pi} \sin^2 \frac{\pi}{2}} \; \frac{1}{\pi}\left(e^{-j\pi} - e^{j\pi}\right) \times j2$

$= \frac{4}{\pi} \; \frac{e^{j\pi} - e^{-j\pi}}{j2}$

$= \frac{4}{\pi} \sin \pi$

$\cos\pi + j \sin\pi$

$$c(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi k F_c t}$$

Odd k

$$= \left(C_1 e^{j2\pi F_c t} + C_{-1} e^{-j2\pi F_c t}\right) + \dots$$

$$= \left(\frac{1}{\pi} j2 \cdot \sin^2\left(\frac{\pi}{2}\right) e^{-j\pi} \times e^{j2\pi F_c t}\right) + \left(\frac{1}{-\pi} \times j2 \times \sin^2\left(\frac{-\pi}{2}\right) e^{+j\pi} \times e^{-j2\pi F_c t}\right)$$

$$= -\frac{j2}{\pi} e^{j2\pi F_c t} + \frac{j2}{\pi} e^{-j2\pi F_c t}$$

$$= \frac{2 \times 2}{\pi}\left\{\frac{e^{j2\pi F_c t} - e^{-j2\pi F_c t}}{j \times 2}\right\} = \frac{4}{\pi} \sin(2\pi F_c t)$$

$$C_3 e^{j2\pi(3F_c)t} + C_{-3} e^{-j2\pi(3F_c)t}$$

$$= \frac{2}{j\pi(3)} e^{j2\pi(3F_c)t} + \frac{2}{j\pi(-3)} e^{-j2\pi(3F_c)t}$$

$$= \frac{2\times 2}{3\pi} \frac{e^{j2\pi(3F_c)t} - e^{-j2\pi(3F_c)t}}{j\times 2}$$

$$= \frac{4}{3\pi} \sin\left(2\pi(3F_c)t\right)$$

$$\frac{4}{\pi} \sin(2\pi F_c t) + \frac{4}{3\pi} \sin\left[2\pi(3F_c)t\right] + \cdots$$

$$= \frac{4}{1}$$

$$C_k = \frac{1}{T}\left\{ \int_0^{T/2} e^{-j2\pi k F_c t}\, dt - \int_{T/2}^{T} e^{-j2\pi k F_c t}\, dt \right\}$$

$$= \frac{1}{T}\left\{ \frac{e^{-j2\pi k F_c t}}{-j2\pi k F_c}\Bigg|_0^{T/2} - \frac{e^{-j2\pi k F_c t}}{-j2\pi k F_c}\Bigg|_{T/2}^{T} \right\}$$

$$= \frac{1}{-j2\pi k F_c T}\left\{ \left(e^{-j2\pi k F_c \frac{T}{2}} - e^{0}\right) - \left(e^{-j2\pi k F_c \times T} - e^{-j2\pi k F_c \frac{T}{2}}\right) \right\}$$

$$= \frac{1}{-j2\pi k F_c T}\left\{ \left(e^{-j\pi k} - 1\right) - \left(e^{-j2\pi k} - e^{-j\pi k}\right) \right\}$$

$$= \frac{1}{j2\pi k F_c T}\left\{ 1 - 2e^{-j\pi k} + e^{-j2\pi k} \right\}$$

$$= \frac{1}{j2\pi k} \left(1 - e^{-j\pi k}\right)^2 = \begin{cases} 0, & k: \text{Even} \\ \frac{2}{j\pi k} & k: \text{Odd} \end{cases}$$

$$C_k = \begin{cases} 0, & k: \text{even} \\ \frac{2}{j\pi k}; & k: \text{odd} \end{cases} \qquad \underline{C_0 = 0}$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+j2\pi k F_c t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{2}{j\pi k} e^{j2\pi k F_c t}$$

$$\frac{2}{j\pi} e^{j2\pi F_c t} - \frac{2}{j\pi} e^{-j2\pi F_c t}$$

$$= \frac{2\times 2}{j\pi} \frac{e^{j2\pi F_c t} - e^{-j2\pi F_c t}}{2}$$

$$= \frac{4}{\pi} \sin(2\pi F_c t)$$

QCM (Qudrature Carrier Multiplexing)

Consider 2 Carrier Signals (Phase Difference = $\pi/2$) $\Rightarrow$ Carriers are in quadrature.

$\cos(2\pi F_c t)$ $\quad$ $\sin(2\pi F_c t)$

$$\sin(2\pi F_c t + \pi/2) = \cos(2\pi F_c t)$$

$$\boxed{x(t) = A_c\, m_I(t) \cos(2\pi F_c t) - A_c\, m_Q(t) \sin(2\pi F_c t)}$$

2 message signals : $m_I(t)$, $m_Q(t)$

$m_I(t)$ is the message, modulated by $\cos(2\pi F_c t)$ or Inphase Carrier

$m_Q(t)$ is the message, modulated by $\sin(2\pi F_c t)$ or Qudrature carrier.

$\Rightarrow$ 2 messages are multiplexed on Qudrature carriers, hence this scheme is termed as Quadrature Carrier multiplexing (QCM)

$$\cos(2\pi F_c t) \cdot \sin(2\pi F_c t) = \frac{1}{2}\sin(4\pi F_c t)$$

$$\downarrow \text{LPF}$$

$$0$$

Therefore $\cos(2\pi F_c t)$ & $\sin 2\pi F_c t$ are orthogonal carriers

This Orthogonality can be used to recover $m_I(t)$, $m_Q(t)$ at Receiver

Demodulation with $\cos(2\pi F_c t)$

$$x(t)\cdot\cos(2\pi F_c t) = \left(A_c m_I(t)\cos 2\pi F_c t - A_c m_Q(t)\sin 2\pi F_c t\right)\cos 2\pi F_c t$$

$$= A_c m_I(t)\cos^2 2\pi F_c t - A_c m_Q(t)\sin 2\pi F_c t\cdot\cos 2\pi F_c t$$

$$= \frac{A_c m_I(t)}{2}(1+\cos 4\pi F_c t) - \frac{A_c m_Q(t)}{2}\sin(4\pi F_c t)$$

$$= \underbrace{\frac{A_c m_I(t)}{2}}_{\text{Baseband Component}} + \underbrace{\frac{A_c m_I(t)}{2}\cos 4\pi F_c t}_{\pm 2F_c} - \underbrace{\frac{A_c m_Q(t)}{2}\sin(4\pi F_c t)}_{+2F_c}$$

$$\downarrow \text{LPF}$$

$$\frac{A_c m_I(t)}{2} \leftarrow \text{Inphase Signal is recovered.}$$

Demodulate ~~$m_Q(t)$~~ with $\sin(2\pi F_c t)$

$$x(t)\cdot\sin(2\pi F_c t) = A_c m_I(t)\cos(2\pi F_c t)\sin(2\pi F_c t) - A_c m_Q(t)\sin^2(2\pi F_c t)$$

$$= \frac{A_c m_I(t)}{2}\sin(4\pi F_c t) - \frac{A_c m_Q(t)}{2}(1-\cos 4\pi F_c t)$$

$$= \frac{A_c m_I(t)}{2}\sin(4\pi F_c t) - \frac{A_c m_Q(t)}{2} + \frac{A_c m_Q(t)}{2}\cos(4\pi F_c t)$$

$$\downarrow \text{LPF}$$

$$-\frac{A_c m_Q(t)}{2} \leftarrow \text{Quadrature message signal is recovered}$$

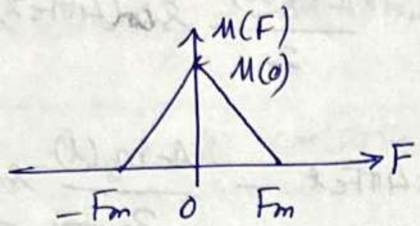
$\cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

SINGLE SIDEBAND MODULATION (SSB)

Consider a message signal $m(t)$

$$m(t) \xleftrightarrow{F.T.} M(F)$$



One sided bandwidth

$\boxed{BW = F_m}$ of $m(t)$

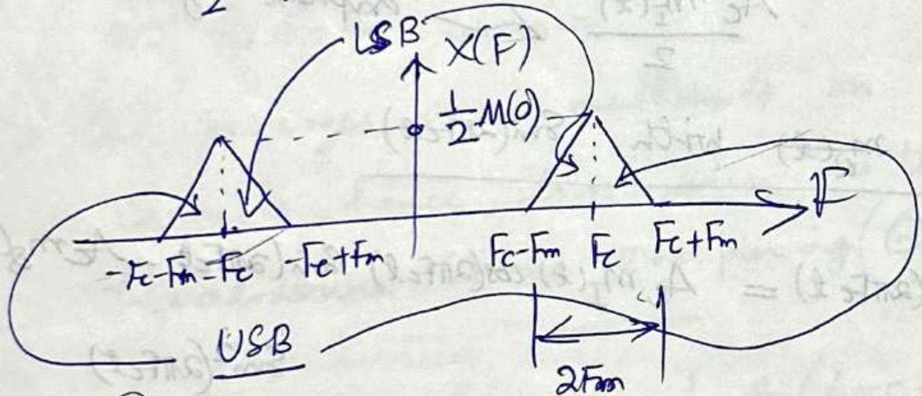
Consider the modulated signal

$$x(t) = m(t) \cdot \cos(2\pi F_c t)$$

$F_c \rightarrow$ Carrier Frequency

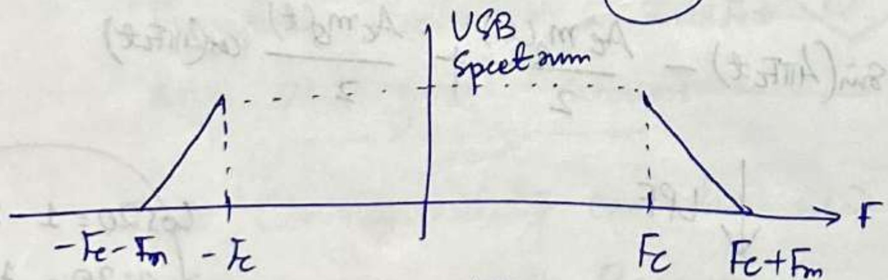
$$X(F) = M(F) * \frac{1}{2}\{\delta(F-F_c) + \delta(F+F_c)\}$$

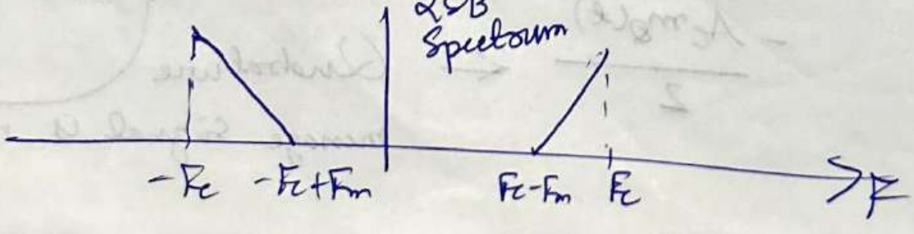
$$= \frac{1}{2}\{M(F-F_c) + M(F+F_c)\}$$



Passband bandwidth of the modulated signal

$= 2F_m$





Advantage of Single band

LSB, USB : Pass-band bandwidth = F_m

Passband bandwidth of LSB/USB = F_m

$\Rightarrow \frac{1}{2} \times$ Bandwidth of DSB-SC

$\Rightarrow$ Spectral efficiency increases.

$\simeq \frac{\text{Information} \ \cancel{\text{given}}}{\text{Bandwidth}}$

Since information of SSB is same as DSB

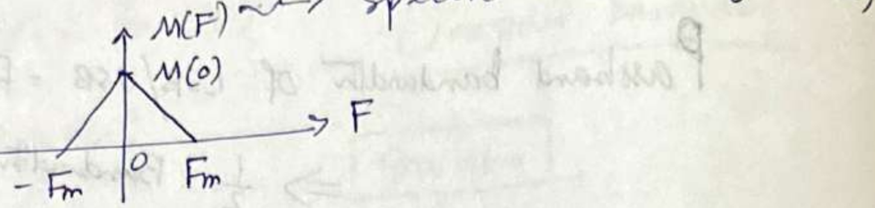
Bandwidth of SSB = $\cancel{\frac{1}{2}}$ ~~Bap~~ ~~Bw of~~ $\frac{1}{2}$ BW of DSB.

$$\boxed{\text{Spectral efficiency of SSB} = 2\times \text{Spectral efficiency of DSB}}$$

4/10/2023

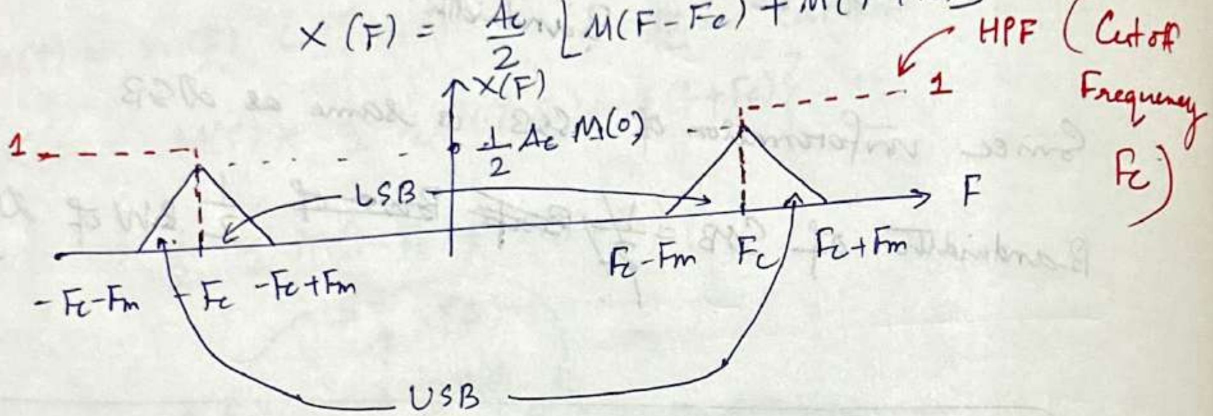
Generation of SSB Modulated Signals

Frequency DISCRIMINATION

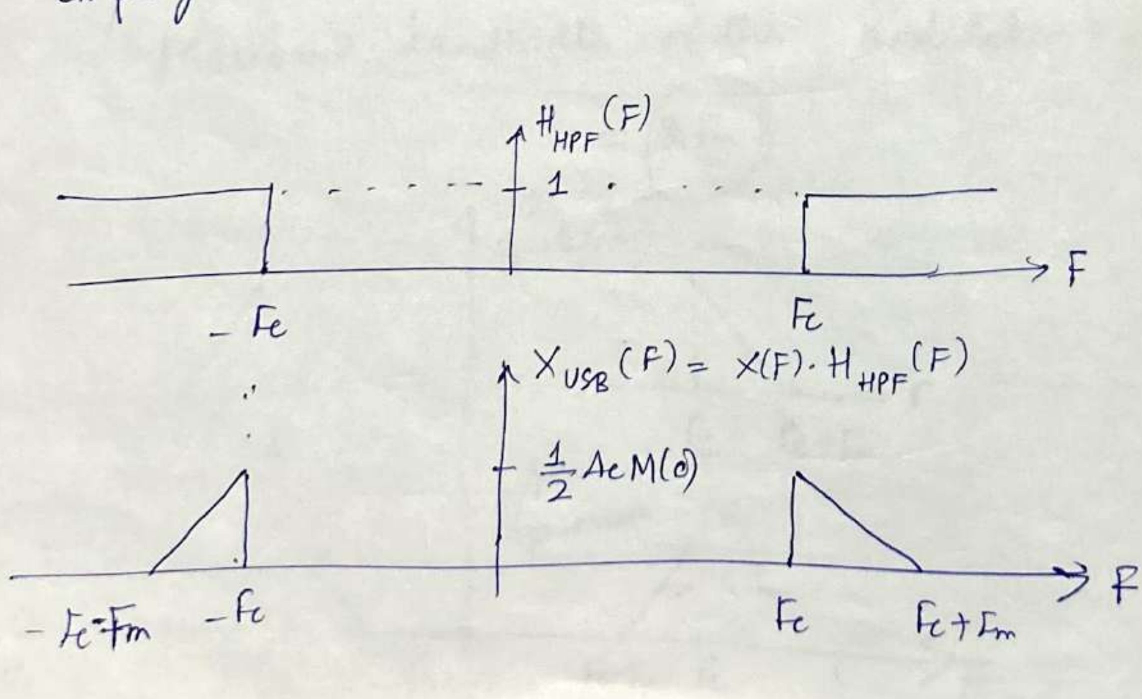


$$x(t) = \underbrace{A_c m(t) \cdot \cos(2\pi F_c t)}_{\text{DSB-SC signal}}$$

$$X(F) = \frac{A_c}{2}\left[M(F-F_c) + M(F+F_c)\right]$$

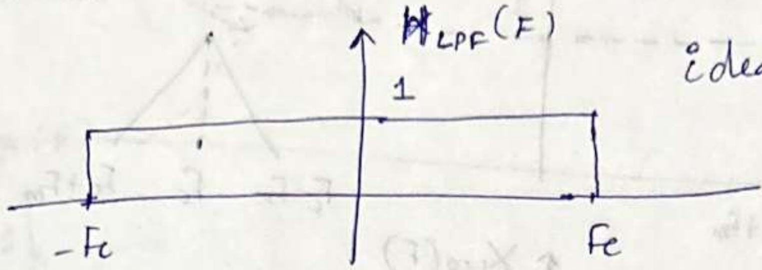


Since LSB & USB are seperated in the frequency domain ⇒ Frequency Discrimination or Filtering can be employed to extract either LSB or USB.

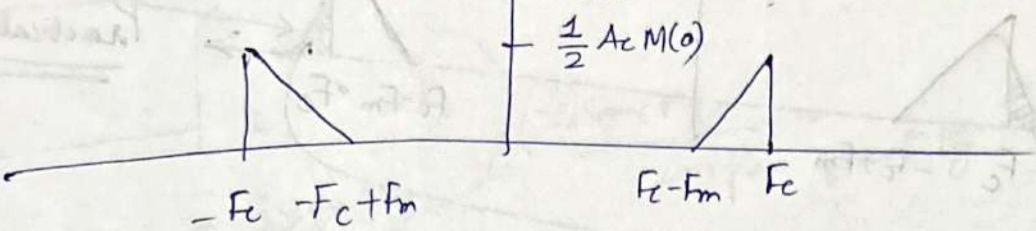


LSB Generation

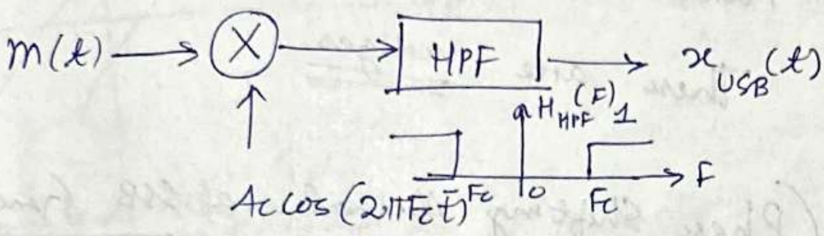
ideal Low Pass Filter



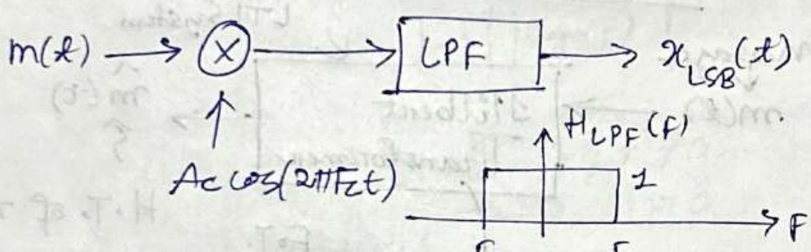
$X_{LSB}(F) = X(F) \cdot H_{LPF}(F)$



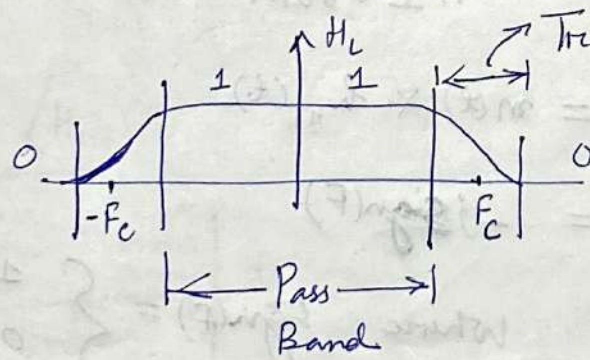
USB Generation



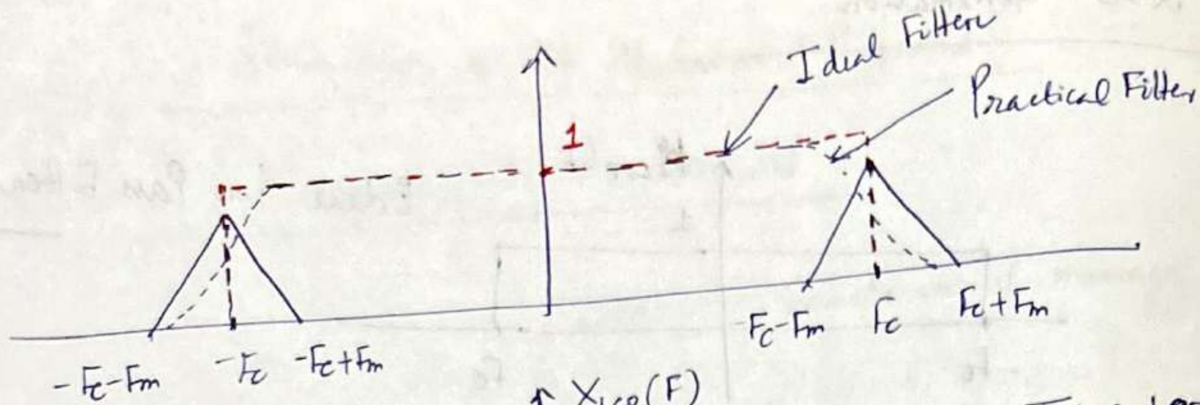
LSB Generation

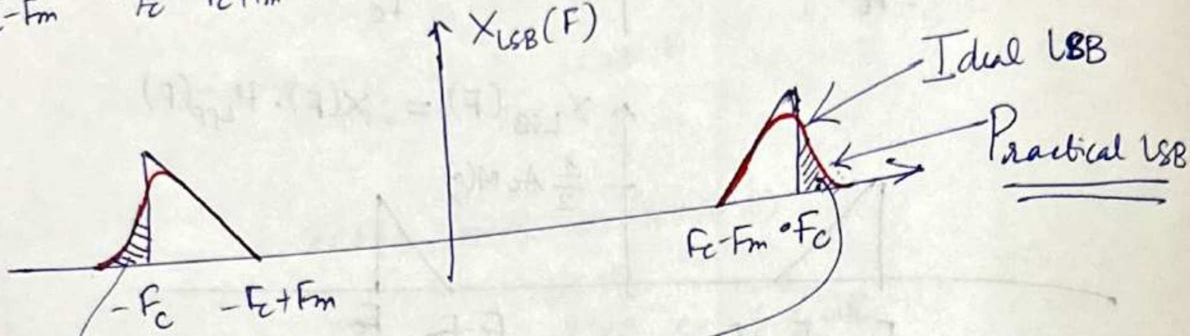


In practice, it is very difficult to design filters with such sharp cut off.



Practical Filters have finite transition bands.



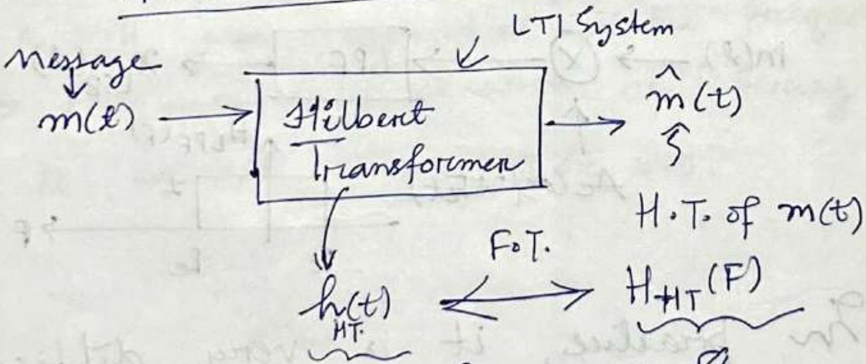


Parts of bands $<-F_c$ & $>F_c$

These are vestiges

(Phase Shifting Method of SSB Generation)

Hilbert Transform (Time Domain Transform)



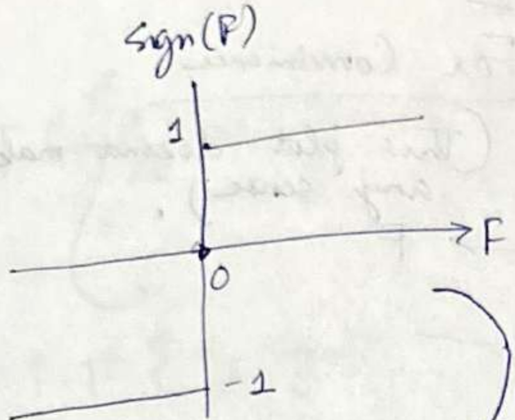
Impulse Response of HILBERT TRANSFORMER

$$\hat{m}(t) = m(t) * h_{HT}(t)$$

$$H_{HT}(F) = -j\,\text{sign}(F)$$

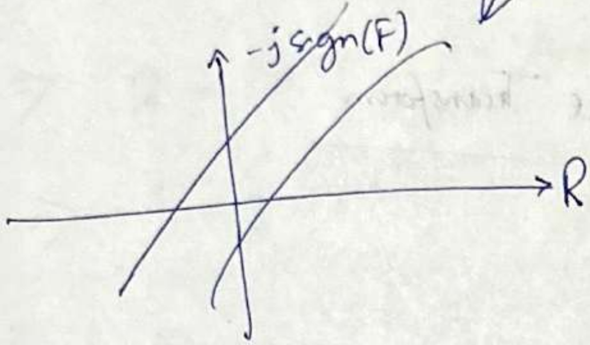
where $\text{sign}(F) = \begin{cases} 1, & F>0 \\ 0, & F=0 \\ -1, & F<0 \end{cases}$

$$H_{HT}(F) = \begin{cases} -j, & F > 0 \\ 0, & F = 0 \\ j, & F < 0 \end{cases}$$



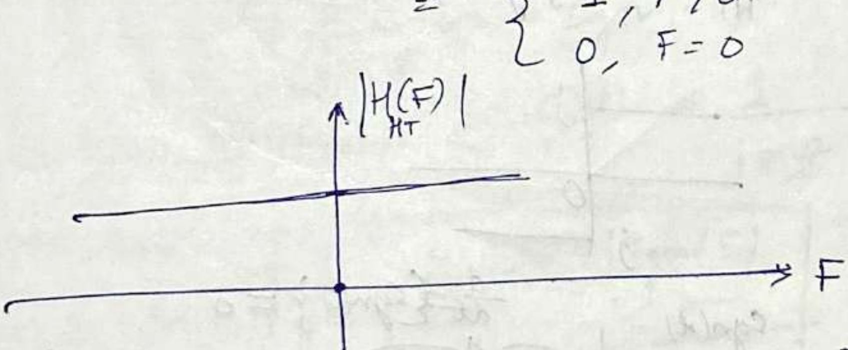
$(-j)\,\mathrm{sgn}(F)$

complex quantity

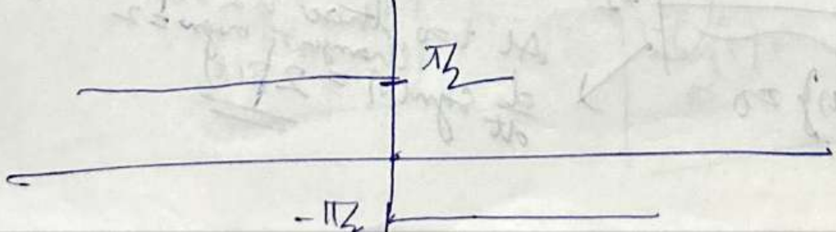


Magnitude Response :

$$|H_{HT}(F)| = |-j\,\mathrm{sgn}(F)|$$
$$= |-j|\,|\mathrm{sgn}(F)|$$
$$= \begin{cases} 1, & F \neq 0 \\ 0, & F = 0 \end{cases}$$



$$\angle H_{HT}(F) = \angle -j\,\mathrm{sgn}(F) = \begin{cases} -\frac{\pi}{2}, & F > 0 \\ 0, & F = 0 \\ \frac{\pi}{2}, & F < 0 \end{cases}$$

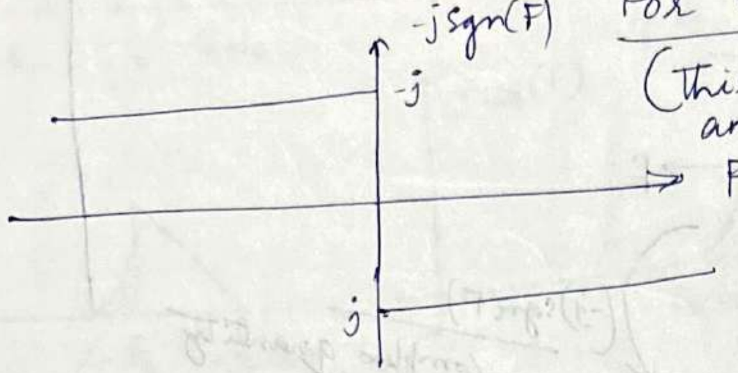


HT is shifting the Phase

All +ve frequencies by $-\frac{\pi}{2}$

All -ve frequencies by $\frac{\pi}{2}$

For Convinience



(This plot doesnot make any sense).

Impulse Response of Hilbert Transform

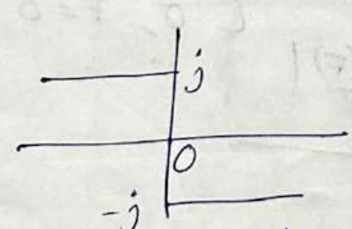
$$h_{HT}(t)$$

Derivative Property of F.T.

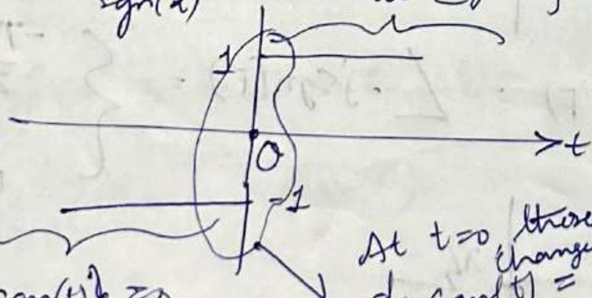
$$x(t) \longleftrightarrow X(F)$$

$$\frac{dx(t)}{dt} \longleftrightarrow j2\pi F X(F)$$

$$H_{HT}(F) = -j\,sgn(F)$$



$\frac{d}{dt}\{sgn(t)\} = 0$



$$sgn(t) = \begin{cases} 1, & t>0 \\ 0, & t=0 \\ -1, & t<0 \end{cases}$$

$\frac{d}{dt}\{sgn(t)\} = 0$

At t=0, there is a step change of magn = 2

$$\frac{d}{dt} sgn(t) = \underline{\underline{2\delta(t)}}$$

$$\frac{d}{dt}\{sgn(t)\} = 2\delta(t)$$

$$F.T \text{ of } \frac{d}{dt}\{sgn(t)\} = FT\{2\delta(t)\} = 2$$

$$FT\left\{\frac{d}{dt}x(t)\right\} = j2\pi F X(F)$$

$$F.T.\left\{\frac{d}{dt}sgn(t)\right\} = j2\pi F \; F.T\{sgn(t)\}$$

$$\Rightarrow 2 = j2\pi F \; FT\{sgn(t)\}$$

$$\Rightarrow FT\{sgn(t)\} = \frac{1}{j\pi F}$$

$$sgn(t) \longleftrightarrow \frac{1}{j\pi F}$$

<u>Duality</u>: $x(t) \longleftrightarrow X(F)$

$$\Rightarrow X(t) \longleftrightarrow x(-F)$$

$$\Rightarrow \frac{1}{j\pi t} \longleftrightarrow sgn(-F)$$

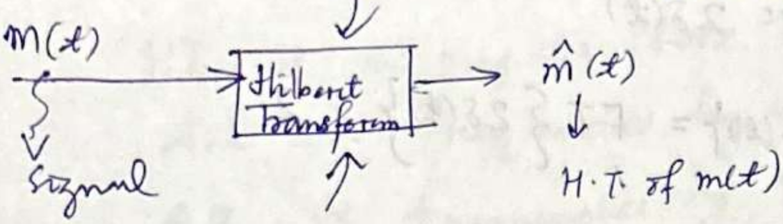
$$\Rightarrow -sgn(F) \longleftrightarrow \frac{1}{j\pi t}$$

$$\Rightarrow \boxed{\frac{1}{\pi t} \longleftrightarrow \underbrace{-j\,sgn(F)}_{H_{HT}(F)}}$$

$$\boxed{h_{HT}(t)}$$

$$\boxed{h_{HT}(t) = \frac{1}{\pi t}}$$

Phase Shifter → Phase of +ve freq by $-\pi/2$
Phase of -ve freq by $\pi/2$



$$H_{HT}(F) = -j\,\mathrm{sgn}(F)$$

$$h_{HT}(t) = \frac{1}{\pi t}$$

$$\boxed{\hat{m}(t) = m(t) * \frac{1}{\pi t}}$$

Product

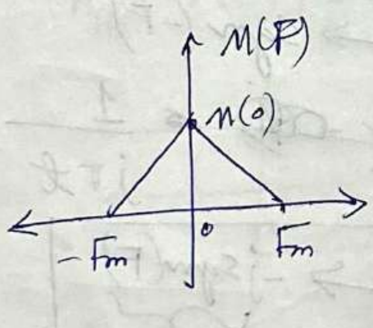
$$\hat{M}(F) = M(F) \cdot [-j\,\mathrm{sgn}(F)]$$

$$= -j\,\mathrm{sgn}(F) M(F)$$

SSB Generation Based on Phase Shifting method

Employs the Hilbert Transform.

Consider message signal m(t) with spectrum M(F)

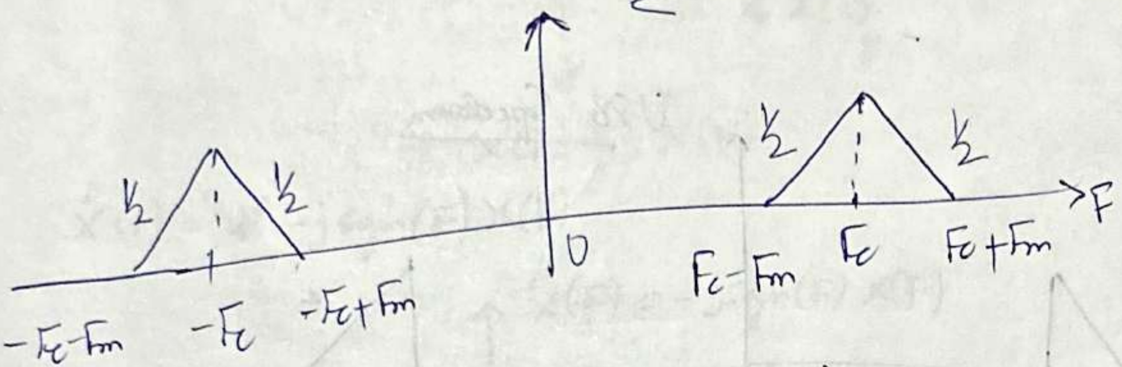


Consider the modulated signal $x(t) = m(t) \cdot \cos(2\pi F_c t) - \hat{m}(t) \sin(2\pi F_c t)$

Where $\hat{m}(t) \longleftrightarrow HT\{m(t)\}$

Since were are using $m(t)$ & the phase shifted signal $\hat{m}(t)$ this is known as Phase Shifting method of generation of SSB.

$$m(t) \cos(2\pi F_c t) \longleftrightarrow \frac{1}{2}\left[M(F-F_c) + M(F+F_c)\right]$$



$$\hat{m}(t) \sin(2\pi F_c t) = \hat{m}(t)\left\{\frac{e^{j2\pi F_c t} - e^{-j2\pi F_c t}}{j2}\right\}$$

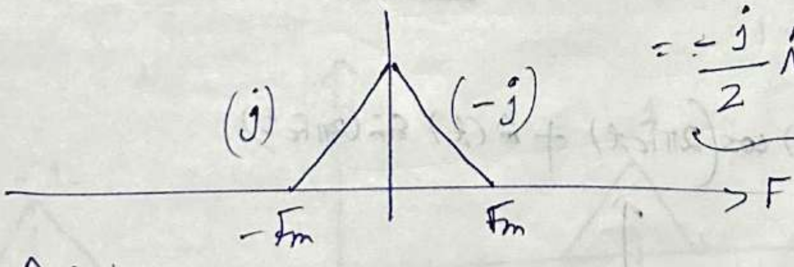
$\hat{m}(t) \longleftrightarrow \hat{M}(F)$

$\hat{M}(F) = M(F)\{-j\,\mathrm{sgn}(F)\}$

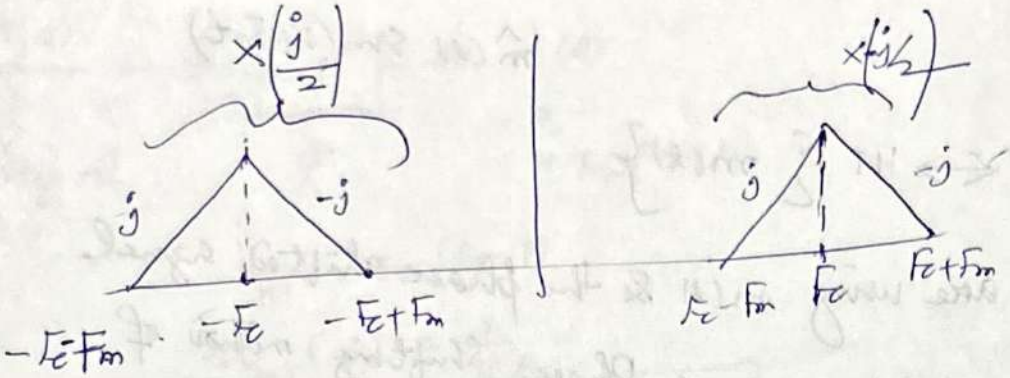
$$\frac{\hat{M}(F)}{2j} * \{\delta(F-F_c) - \delta(F+F_c)\}$$

$$= -\frac{j}{2}\hat{M}(F-F_c) + \frac{j}{2}\hat{M}(F+F_c)$$

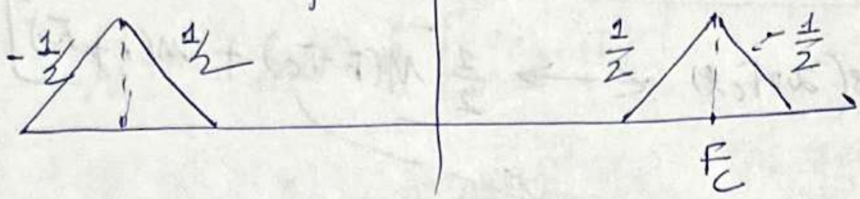
Spectrum of $\hat{m}(t) \sin(2\pi F_c t)$



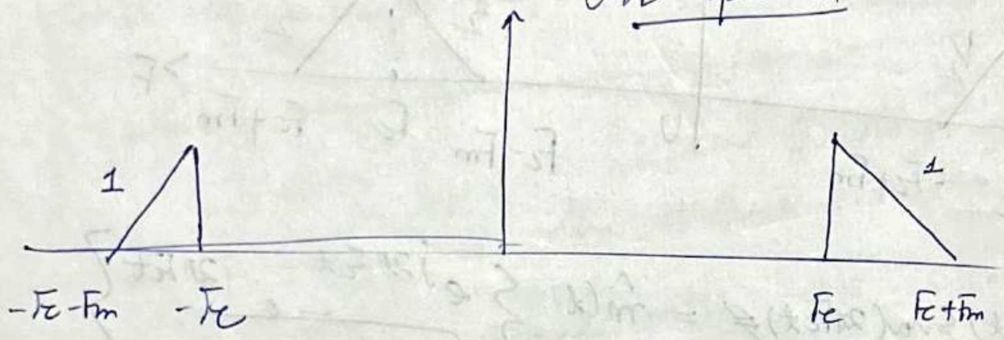
$\hat{m}(t)$:



Spectrum of $\hat{m}(t).\sin(2\pi F_c t)$



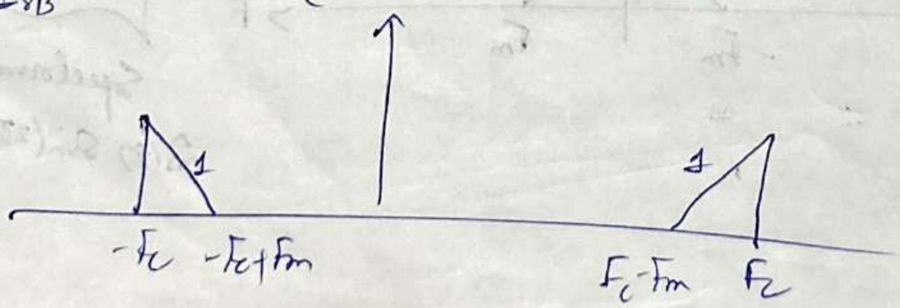
USB Spectrum



$$x_{USB}(t) = m(t)\cos(2\pi F_c t) - \hat{m}(t)\sin(2\pi F_c t)$$

Upper Sideband Modulated Signal

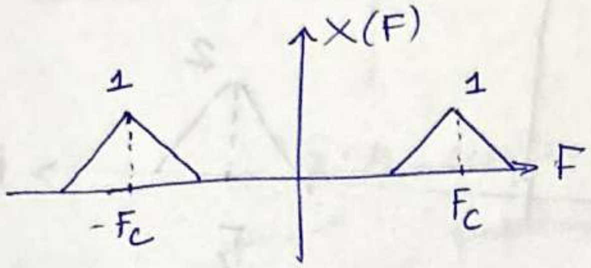
$$x_{LSB}(t) = m(t)\cos(2\pi F_c t) + \hat{m}(t)\sin(2\pi F_c t)$$



Complex Envelope

Consider a passband signal $x(t)$

$$x(t) \longleftrightarrow X(F)$$



Consider a new signal,

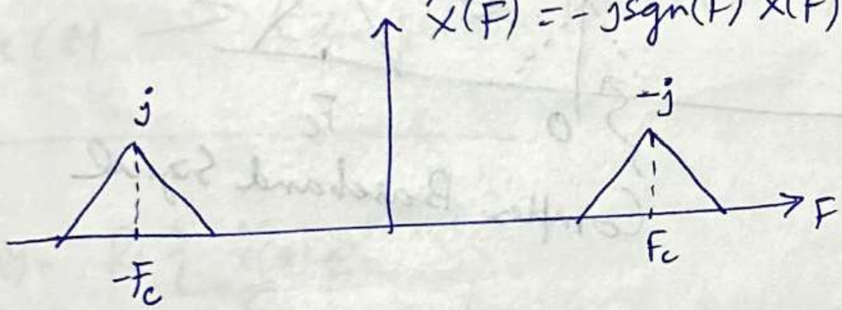
$$x(t) + j\underbrace{\hat{x}(t)}_{\text{H.T. of } x(t)}$$

$$\Big\updownarrow$$

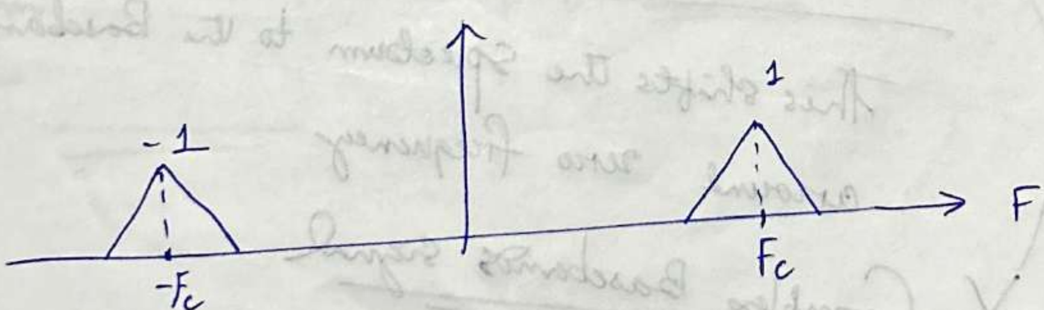
$$X(F) + j\hat{X}(F)$$

$$\hat{X}(F) = \cancel{H} -j\,\mathrm{sgn}(F).X(F)$$

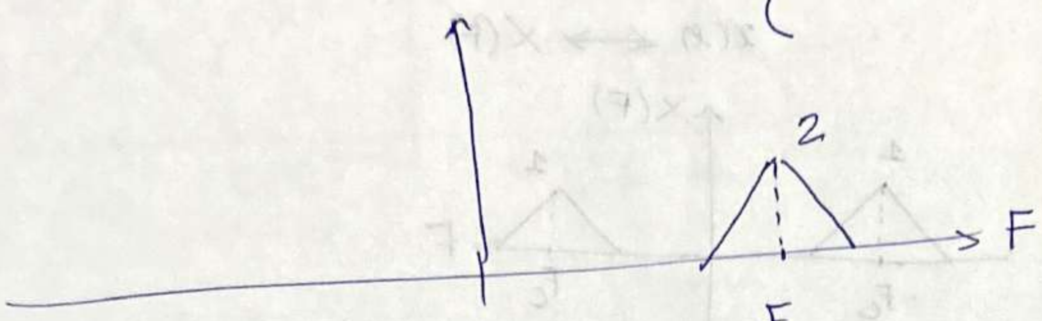
$$\hat{X}(F) = -j\,\mathrm{sgn}(F)\,X(F)$$



$$j\hat{X}(F)$$



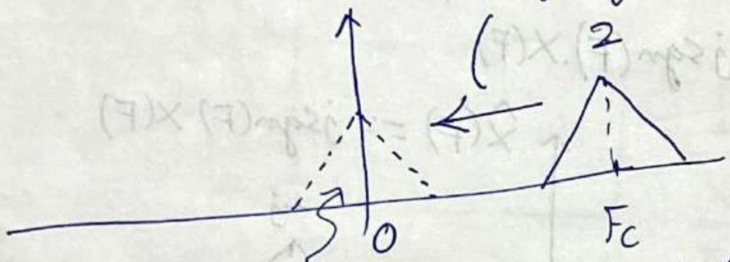
$$X(F)+j\hat{X}(F) = \begin{cases} 2X(F), & F>0 \\ 0, & F<0 \end{cases}$$



$$\underbrace{x(t)+j\hat{x}(t)}_{\text{Complex Pre-envelope of } x(t)} \longleftrightarrow \begin{cases} 2X(F), & F>0 \\ 0, & F<0 \end{cases}$$

Complex Pre-envelope of $x(t)$

Shift by F_c to the left



Complex Baseband Signal

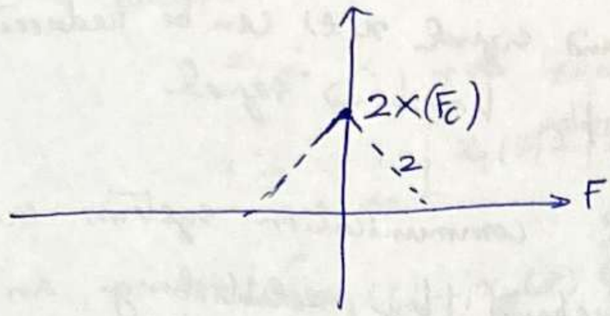
$$\left[x(t)+j\hat{x}(t)\right]\cdot e^{-j2\pi F_c t}$$

This shifts the Spectrum to the Baseband; i.e, around zero frequency

Complex Baseband signal

Complex baseband equivalent signal of passband signal $x(t)$

$\tilde{X}(F)$ = Spectrum of $\tilde{x}(t)$



$\tilde{x}(t)$ = complex baseband equivalent of passband signal $x(t)$.

Complex Envelope of $x(t)$

$$Re\{\tilde{x}(t)e^{j2\pi F_c t}\}$$

$$= Re\{(x(t)+j\hat{x}(t))e^{-j2\pi F_c t} \cdot e^{j2\pi F_c t}\}$$

$$= Re\{x(t)+j\hat{x}(t)\}$$

$$= x(t) \longrightarrow \text{Original passband signal}$$

$$\underbrace{x(t)}_{\text{Passband signal}} = Re\{\underbrace{\tilde{x}(t)e^{j2\pi F_c t}}\}$$

Complex Envelope of $\tilde{x}(t)e^{j2\pi F_c t}$

So,

$\Longrightarrow$ Every passband signal $x(t)$ can be reduced to an equivalent complex baseband signal

$\Rightarrow$ All the analysis of communication systems can be carried out in baseband by establishing an equivalence between ~~basebo~~ passband & complex baseband signal.

$\Rightarrow$ An unified frame work can be developed for analysis of communication systems.

QCM Signal

(Quadrature Carrier Modulation)

$$x(t) = \underbrace{x_I(t)}_{\text{Inphase Signal}} \cos(2\pi F_c t) - \underbrace{x_Q(t)}_{\text{Quadrature Signal}} \sin(2\pi F_c t)$$

$x_I(t)$, $x_Q(t)$ $\leftarrow$ Baseband Signals.

$$x(t) = x_I(t)\left[\frac{e^{j2\pi F_c t} + e^{-j2\pi F_c t}}{2}\right] - x_Q(t)\left[\frac{e^{j2\pi F_c t} - e^{-j2\pi F_c t}}{2j}\right]$$

$$= \frac{1}{2}\underbrace{\{x_I(t) + j\,x_Q(t)\}}_{\text{Baseband}} e^{j2\pi F_c t} + \frac{1}{2}\underbrace{\{x_I(t) - j\,x_Q(t)\}}_{\text{Baseband}} e^{-j2\pi F_c t}$$

Shifts to F_c — Corresponds to +ve frequency Band

Shifts to $-F_c$ — Corresponds to -ve frequency band.

Complex Pre-envelope:

$$= 2 \times \text{+ve Frequency Band Signal}$$

$$= 2 \times \frac{1}{2}\left[x_I(t) + jx_Q(t)\right]e^{j2\pi F_c t}$$

$$= \underbrace{\left\{x_I(t) + jx_Q(t)\right\}e^{j2\pi F_c t}}_{\text{Complex Pre-envelope of QCM signal } x(t).}$$

Complex envelope $\tilde{x}(t)$ is,

$$\tilde{x}(t) = x_p(t)e^{-j2\pi F_c t}$$

$$= \left\{x_I(t) + jx_Q(t)\right\}e^{j2\pi F_c t} \cdot e^{-j2\pi F_c t}$$

$$\boxed{\tilde{x}(t) = x_I(t) + jx_Q(t)}$$

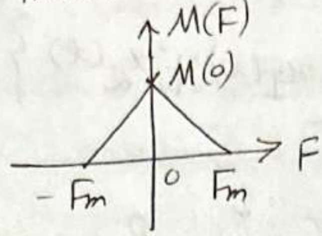
Complex Envelope or Complex Baseband equivalent of QCM signal.

VESTIGIAL SIDEBAND MODULATION

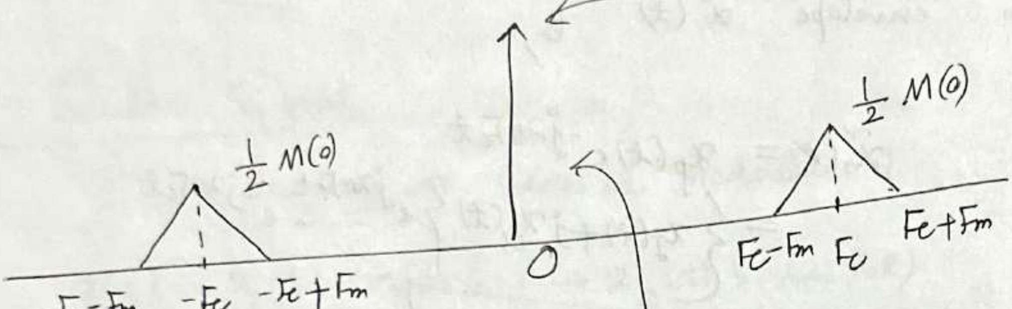
(VSB)

Why?

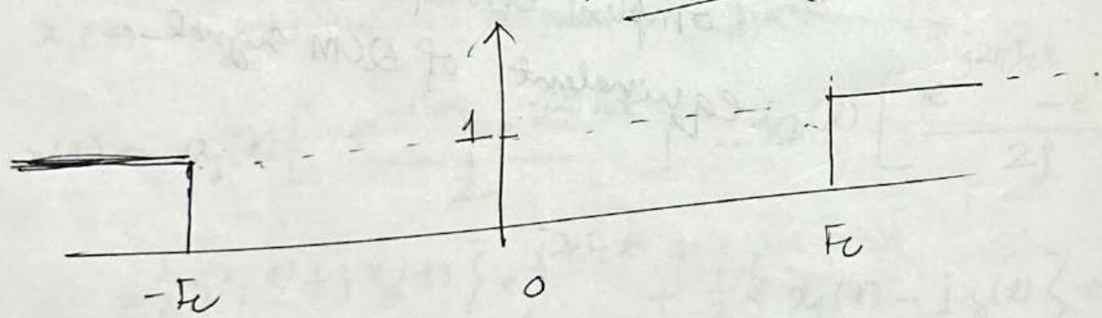
$m(t) \leftarrow$ message.

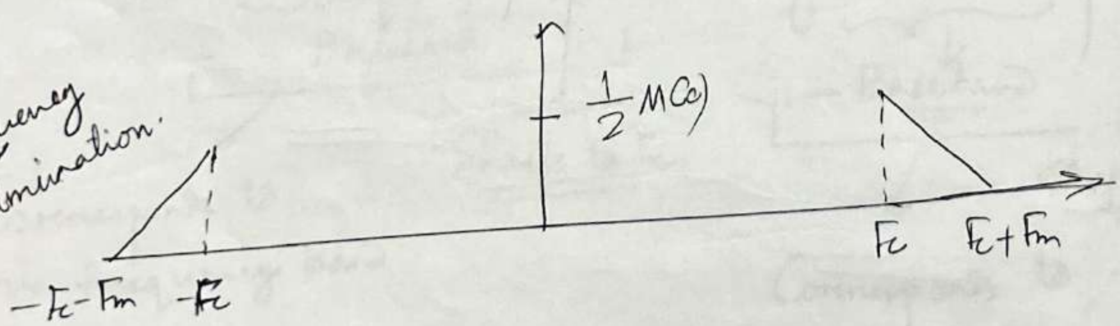


$$m(t)\cdot\cos(2\pi F_c t) \longleftrightarrow \frac{1}{2}\left[M(F-F_c)+M(F+F_c)\right]$$



Spectrum of $m(t)\cdot\cos(2\pi F_c t)$

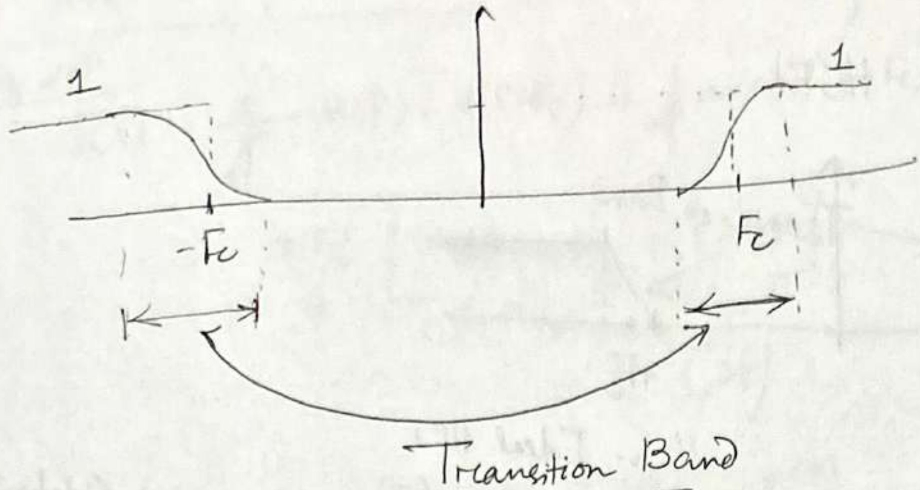


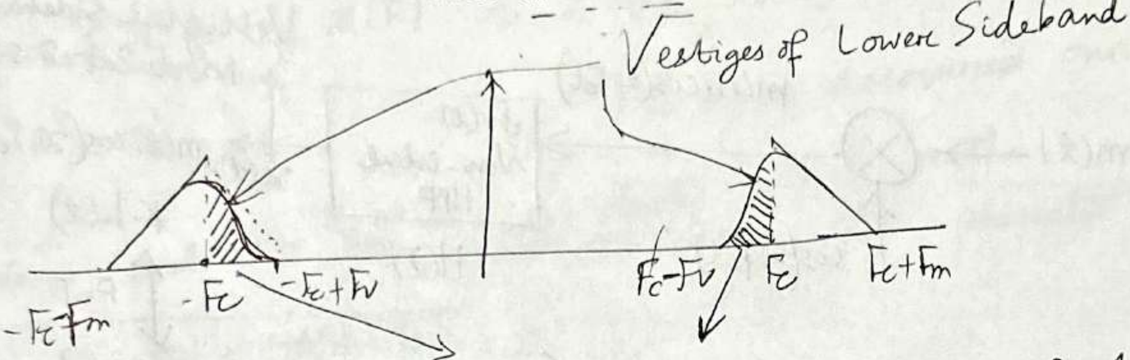


Frequency Discrimination.

for SSB Generation

To design an HPF with such sharp cut off frequency





Because of Vestige requires a small portion of LSB.

F_v :- Bandwidth of vestige.

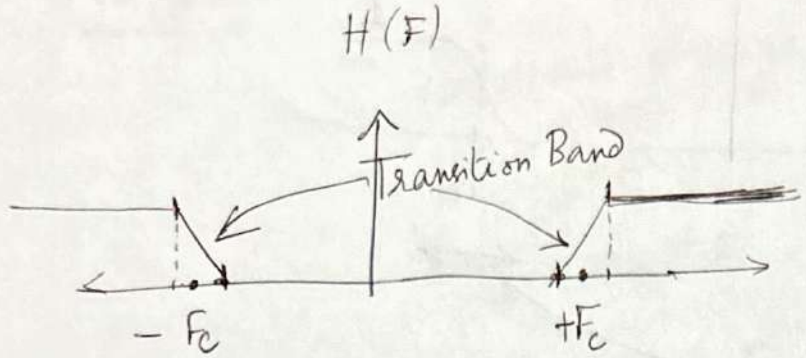
Net bandwidth of the signal $= \underline{\underline{F_m + F_v}}$

Increasing Spectral efficiency of Vestige $\rightarrow$

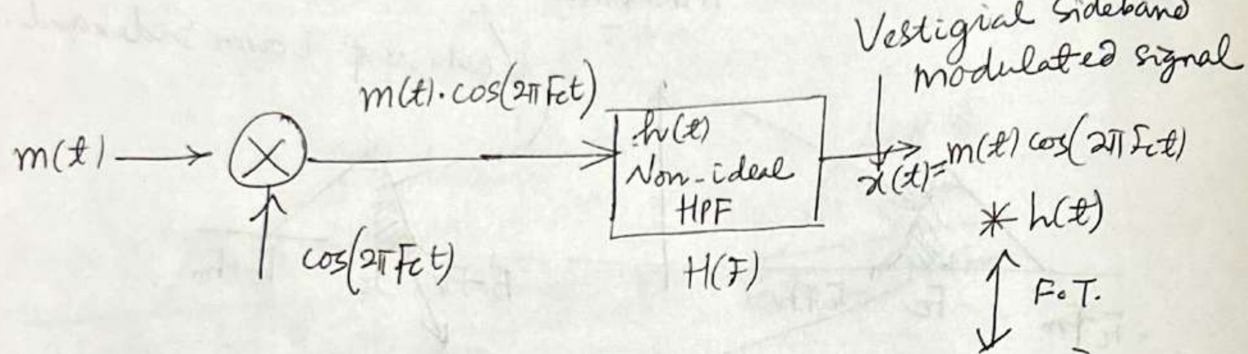
$$\underbrace{DSB\text{-}SC}_{2F_m} > \underbrace{VSB}_{F_m+F_v} > \underbrace{\overbrace{USB / LSB}^{SSB}}_{F_m}$$

VSB has Lower complexity of Implementation in comparison to USB since it doesn't require ideal HPF
(USB + Vestige of LSB)

Properties of VSB Filter

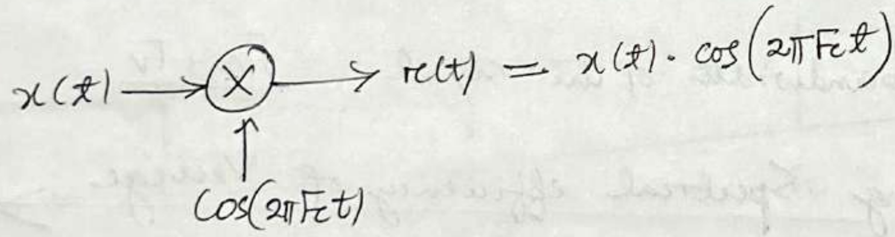


Non- Ideal HPF



$$X(F) = \left\{ \frac{1}{2} M(F-F_c) + \frac{1}{2} M(F+F_c) \right\} H(F)$$

At the Receiver:



$$r(t) = x(t) \cdot \cos(2\pi F_c t)$$

$$R(F) = \frac{1}{2} X(F-F_c) + \frac{1}{2} X(F+F_c)$$

$$= \frac{1}{2} \left\{ \left(\frac{1}{2} M(F-2F_c) + \frac{1}{2} M(F) \right) H(F-F_c) \right\}$$

$$+ \frac{1}{2} \left\{ \left(\frac{1}{2} M(F) + \frac{1}{2} M(F+2F_c) \right) H(F+F_c) \right\}$$

$$= \frac{1}{4} M(F-2F_c) \cdot H(F-F_c) + \frac{1}{4} M(F) H(F-F_c)$$

$$+ \frac{1}{4} M(F) \cdot H(F+F_c) + \frac{1}{4} M(F+2F_c) \cdot H(F+F_c)$$

$M(F-2F_c)$ & $M(F+2F_c)$ can be eliminated by LPF

Pass $r_c(t)$ through an LPF

O/p of LPF
$$\tilde{R}(F) = \frac{1}{4} M(F) \cdot H(F-F_c) + \frac{1}{4} M(F) H(F+F_c)$$

$$= \frac{M(F)}{4} \left[\underbrace{H(F-F_c) + H(F+F_c)}_{= 1 \text{ (if)}} \right]$$

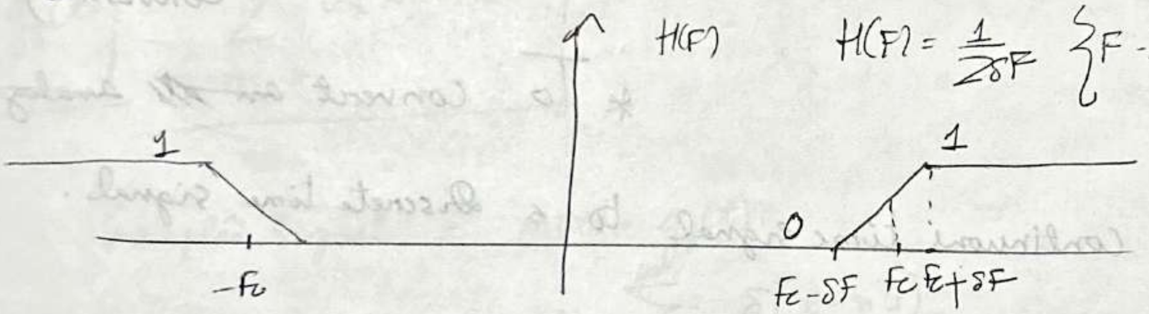
$$\tilde{R}(F) = \frac{1}{4} M(F) \Rightarrow \tilde{r}(t) = \frac{1}{4} m(t)$$

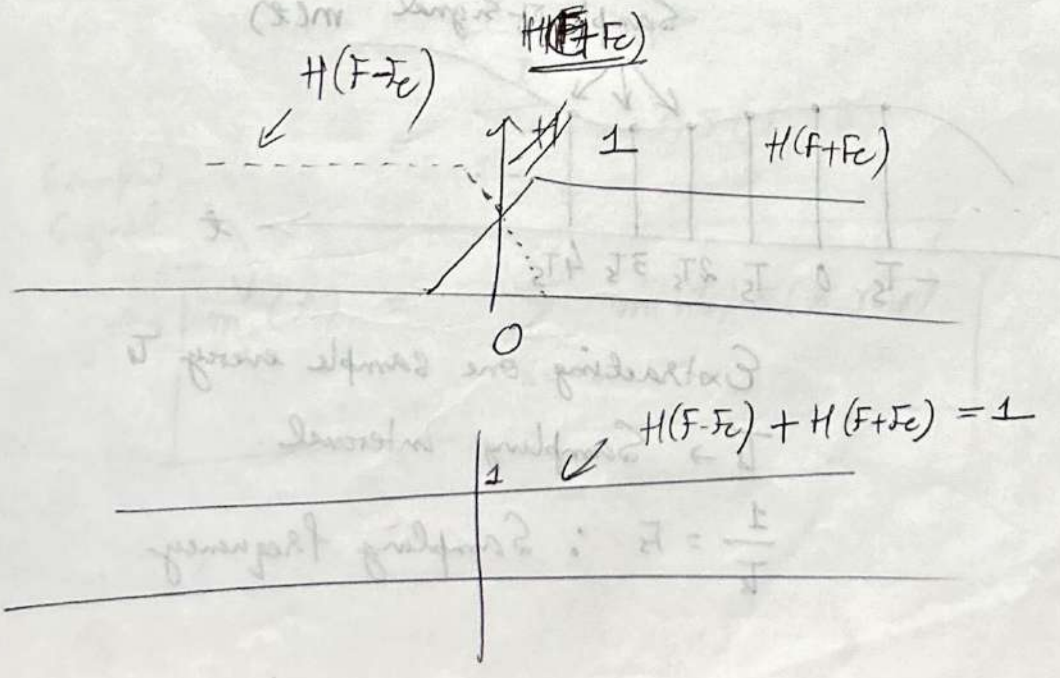
$\Rightarrow$ We have recovered our original signal $m(t)$.

Non-ideal VSB filter must satisfy the property

$$\boxed{H(F-F_c) + H(F+F_c) = 1}$$

$$H(F) = \frac{1}{2\delta F} \{F - F_c + \delta F\}$$





If the VSB filter is symmetric about F_c

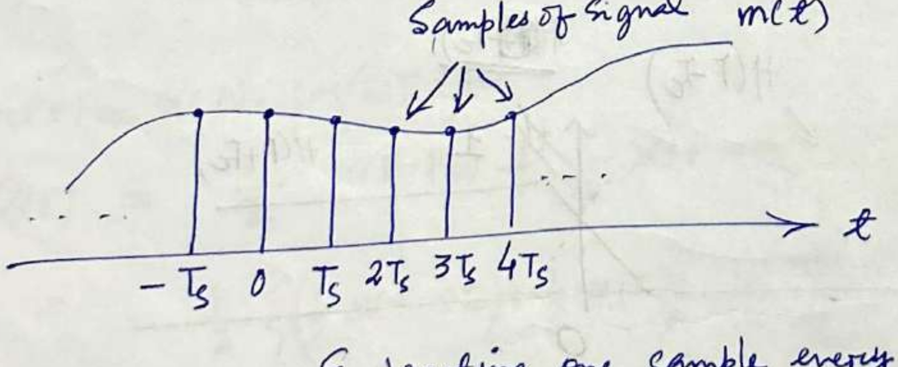
$$H(F) = \begin{cases} \dfrac{1}{2\delta F}\left(F - F_c + \delta F\right), & F_c - \delta F \le F \le F_c + \delta F \\ 1, & F \ge F_c + \delta F \end{cases}$$

A particular non-ideal HPF which satisfies VSB property

Analog to Digital Conversion

1. Sampling (1st step of Analog to Digital Conversion)

* To convert ~~an st~~ ~~analog~~ a continuous time signal to a Discrete time signal.

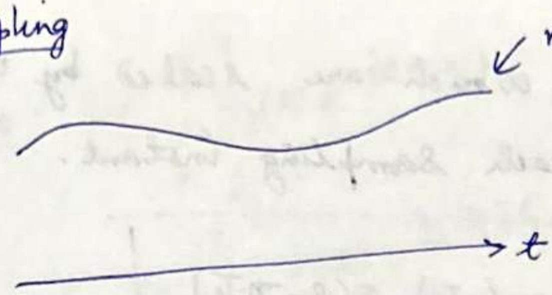


Extracting one sample every T_s

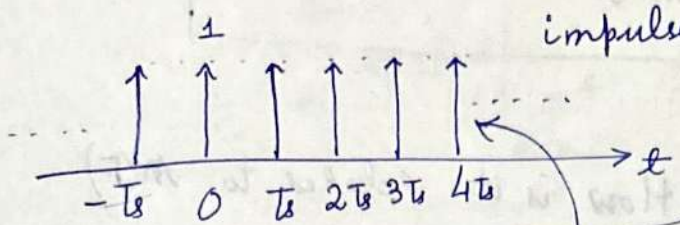
$T_s \rightarrow$ Sampling interval.

$\frac{1}{T_s} = F_s$: Sampling frequency

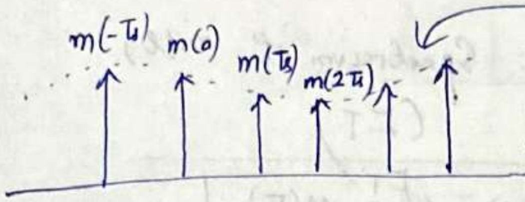
Ideal Sampling



impulse train



Sampled Signal



Periodic Signal of impulses with period $= T_s$.

Impulse train : impulses at $0, \pm T_s, \pm 2T_s, +3T_s \cdots$

$$\cdots + \delta(t+T_s) + \delta(t) + \delta(t-T_s) + \delta(t-2T_s) + \cdots$$

$$\boxed{g_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_s)}$$ ← impulse train

$$\text{Sampled Signal} = m(t) \cdot g_\delta(t)$$

$$= m(t) \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$$

$$= \sum_{n=-\infty}^{\infty} m(t)\,\delta(t-nT_s)$$

Sampled Signal

$$\Rightarrow \boxed{m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\,\delta(t-nT_s)}$$

Sequence of impulse which are scaled by the value of message signal at each sampling instant.

$$\boxed{m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\cdot \delta(t - nT_s)}$$

$\Big\updownarrow$ F.T.

$M_\delta(F) =$ How is it related to $M(F)$

$M(F)$: Spectrum of $m(t)$ (F.T.)

$$\boxed{m(t) \xleftrightarrow{FT} M(F)}$$

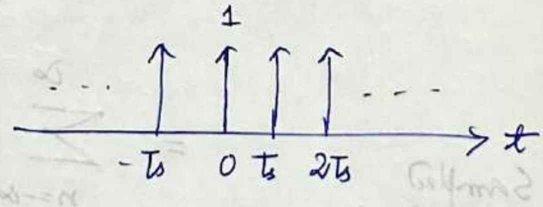
By sampling we lose information $\longrightarrow$ This may lead to distortion.

<u>How to fix</u> the parameters of sampling to minimize or if <u>possible avoid distortion?</u>

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

F.T. $\updownarrow$

$G_\delta(F)$



<u>Periodic signal</u>

with period = T_s.

$$g_\delta(t) = \sum_{k=-\infty}^{\infty} C_k e^{j2\pi k F_s t}$$

$\left(F_s = \frac{1}{T_s}\right)$

$\downarrow$

fundamental frequency = Sampling frequency

$$C_k = \frac{1}{T_s}\int_0^{T_s} g_\delta(t)\, e^{-j2\pi k F_s t}\, dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-j2\pi k F_s t}\, dt \quad \cancel{= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} dt = \frac{1}{T_s}\times T_s = 1}$$

$$= \frac{1}{T_s}\, e^{-j2\pi k F_s t}\Big|_{t=0} = \frac{1}{T_s}$$

$$\therefore \boxed{C_k = \frac{1}{T_s}}$$

$$g_\delta(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{j2\pi k F_s t}$$

$$\Rightarrow \boxed{g_\delta(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{j2\pi k F_s t}}$$

$$\boxed{G_\delta(F) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} \delta(F - kF_s)}$$

Impulse train in frequency domain.

Spacing between impulses $= F_s = \frac{1}{T_s}$

Spacing in time $= T_s$
Spacing in frequency $= \frac{1}{T_s} = F_s$

As spacing in time domain increases
spacing in frequency domain decreases.

Fourier Transform of Sampled Signal

Original Analog Signal

$$m_\delta(t) = m(t) \cdot g_\delta(t)$$

Sampled Analog Signal

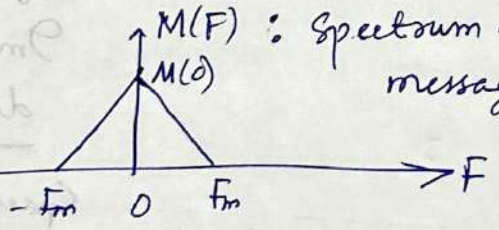
$$M_\delta(F) = M(F) * G_\delta(F)$$

$$= M(F) * \cancel{\tfrac{1}{T_s}} F_s \sum_{k=-\infty}^{\infty} \delta(F - kF_s)$$

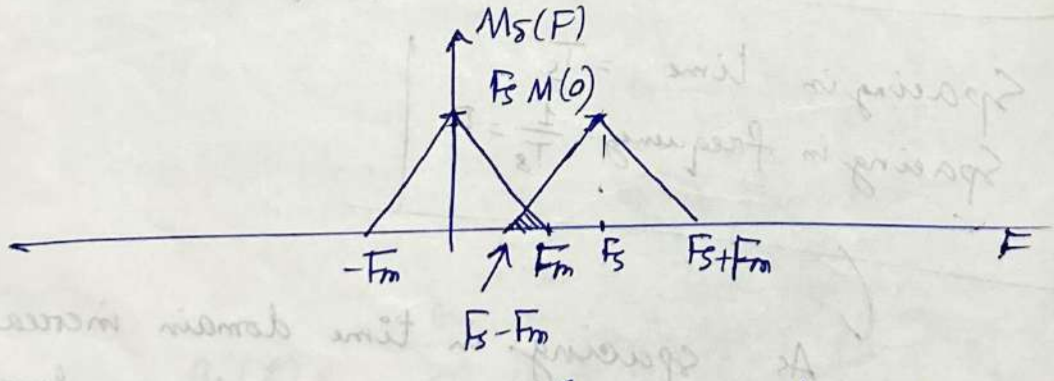
$$\boxed{M_\delta(F) = F_s \sum_{k=-\infty}^{\infty} M(F - kF_s)}$$

Spectrum of Sampled signal = Sum of all copies of the original message spectrum $M(F)$ shifted to every kF_s

k^{th} multiple of F_s



$M(F)$: Spectrum of Original message



The overlap lead to distortion.
This distortion arises when,

$$F_s - F_m < F_m$$
$$\Rightarrow F_s < 2F_m.$$

$\boxed{F_s < 2F_m}$

This distortion which occurs because of copies of spectrum shifted by multiples of ~~kf~~ F_s (kF_s) is termed as aliasing.

Aliasing occurs if

$$\boxed{F_s < 2F_m}$$

Sampling frequency < 2 (Max. ~~message~~ ~~frequen~~ frequency $m(t)$)

To avoid aliasing distortion,

$$\boxed{F_s \geq 2F_m}$$

Nyquist criterion ~~for~~

(Or) " Sampling Theorem.

Sampling & Reconstruction

$m(t) \rightarrow$ Original signal

$$m_\delta(t) = m(t) \times g_\delta(t) = m(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$\uparrow$ impulse train

$F_s = \frac{1}{T_s}$: Sampling frequency.

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\, \delta(t - nT_s)$$

Sum of impulses at nT_s scaled by $m(nT_s)$

$$M_\delta(F) = M(F) * G_\delta(F)$$

$$= M(F) * F_s \sum_{k=-\infty}^{\infty} \delta(F - kF_s)$$

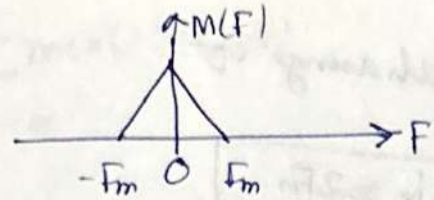
$$\boxed{M_\delta(F) = F_s \sum_{k=-\infty}^{\infty} M(F - kF_s)}$$

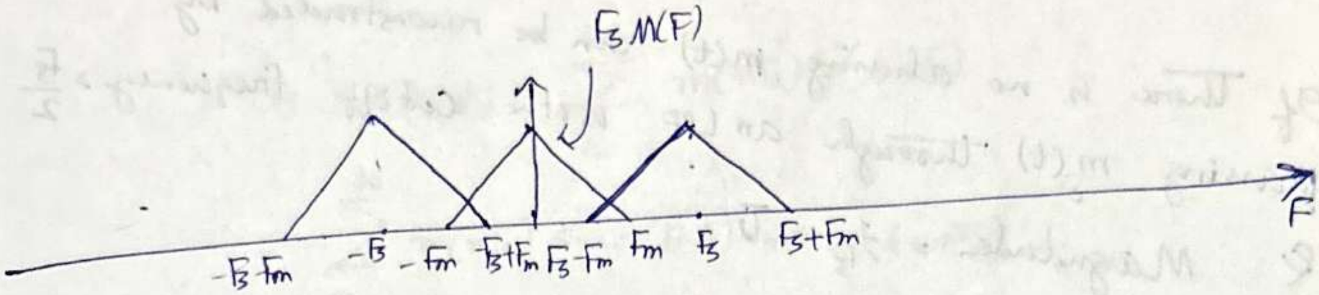
(F.T. of sampled function)

$\downarrow$

$$\underbrace{F_s M(F) + F_s M(F \pm F_s) + F_s M(F \pm 2F_s) + \cdots}$$

Sum of spectral copies of $M(F)$ shifted to each kF_s.





Overlap of spectral copies leads to distortion.

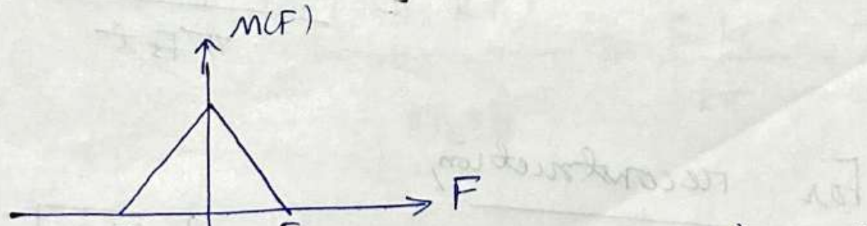
Spectrum of ~~s~~ ... Sampled ~~spectrum~~ signal is a distorted version of original spectrum. → ALIASING

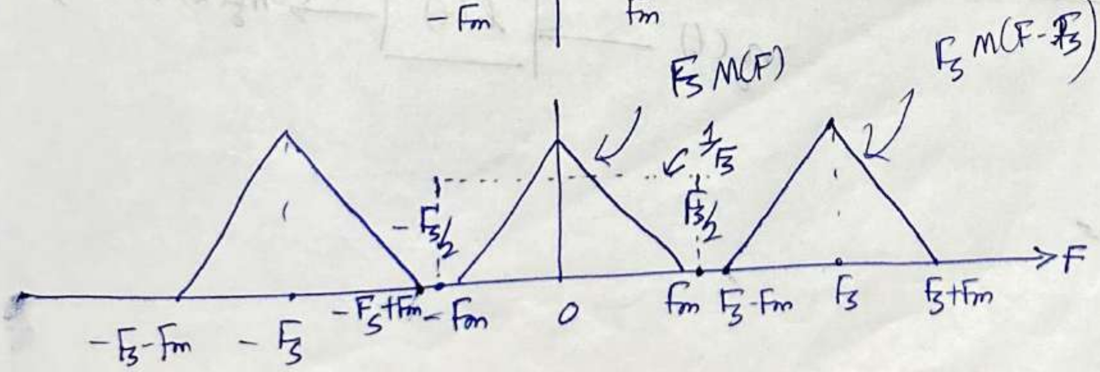
Aliasing occurs if,

$$F_s - F_m < F_m$$

$$\Rightarrow \boxed{F_s < 2F_m}$$

To avoid aliasing, $\boxed{F_s \geq 2F_m}$: Nyquist criterion



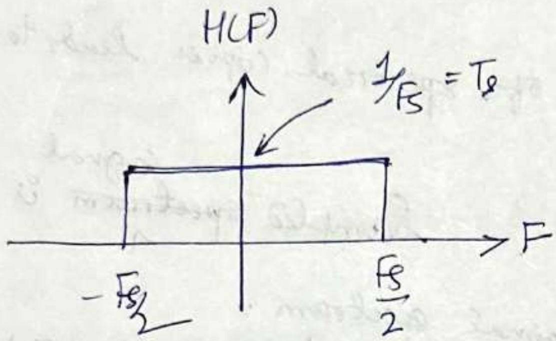


No overlap or aliasing if

$$\boxed{F_s \geq 2F_m}$$

If there is no aliasing, $m(t)$ can be reconstructed by passing $m_\delta(t)$ through an LPF with cut off frequency $= \frac{F_s}{2}$

& Magnitude $= 1/F_s = T_s$



Low pass filter for reconstruction

$$H(F) = \begin{cases} T_s, & |F| \leq \frac{F_s}{2} \\ 0, & \cancel{|F| > \frac{F_s}{2}} \text{ (otherwise)} \end{cases}$$

This can be derived By Duality

$$\boxed{FT[H(F)] \quad h(t) = \text{sinc}(F_s t)}$$

$$= \frac{\sin \pi F_s t}{\pi F_s t}$$

For reconstruction,

$$m_\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow m_\delta(t) * h(t)$$

$$m_\delta(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\, \delta(t - nT_s)$$

$$m(t) = m_\delta(t) * \operatorname{sinc}(F_s t)$$

$$= \operatorname{sinc}(F_s t) * \sum_{n=-\infty}^{\infty} m(nT_s)\, \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} m(nT_s)\, \operatorname{sinc}(F_s t) * \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} m(nT_s)\, \operatorname{sinc}\left[F_s (t - nT_s)\right]$$

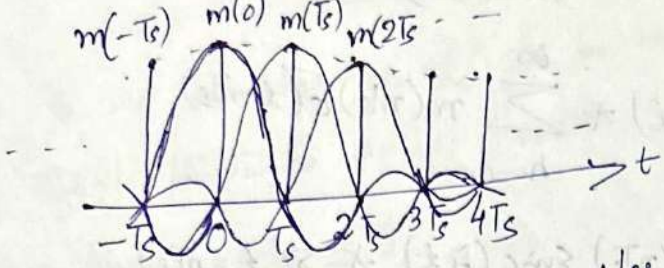
$$\Rightarrow m(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\, \operatorname{sinc}(F_s t - n)$$

Shifting $\operatorname{sinc}(F_s t)$ to each nT_s & scaling by $m(nT_s)$ followed by sum.

$$\operatorname{sinc}(F_s t) = \frac{\sin(\pi F_s t)}{(\pi F_s t)}$$

Zeros are at KT_s

$$\frac{\sin(\pi F_s K T_s)}{\pi F_s K T_s} = \frac{\sin K\pi}{K\pi} = 0$$



interpolating samples using sinc filter.

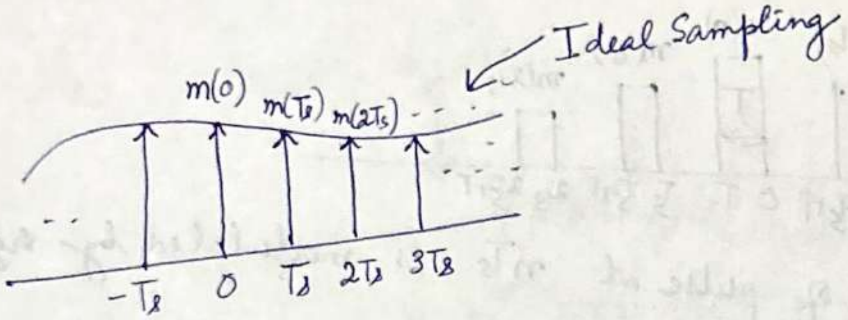
$$m(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\, \mathrm{sinc}\left[F_s(t-nT_s)\right]$$

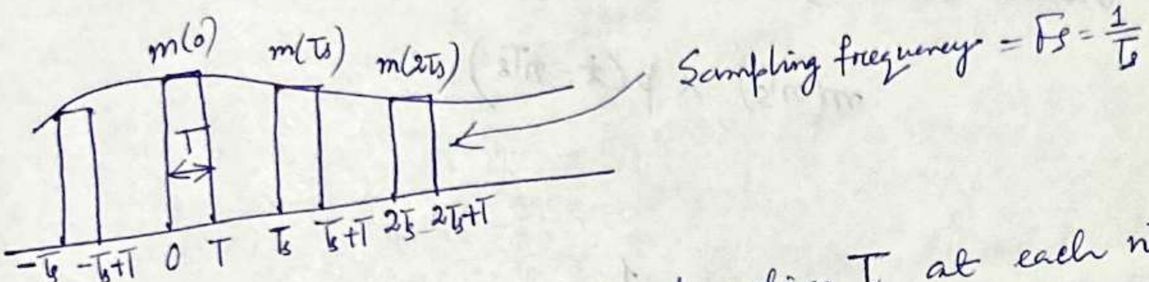
Shift sinc$(F_s t)$ to each nT_s scaled by $m(nT_s)$

sum of all shifted copies of sinc$(F_s t)$ ✓

sinc response : interpolation filter

PAM (Pulse Amplitude Modulation)





Sampling using pulses of duration T at each nT_s.

The n^{th} pulse is from nT_s to $nT_s + T$ & amplitude of the pulse = $m(nT_s)$ {value of signal at nT_s}

{ Sampling at each nT_s & holding the sample value constant for a duration $= T$ }

→ Sample & Hold Operation.

$T << T_s$

Pulse Duration

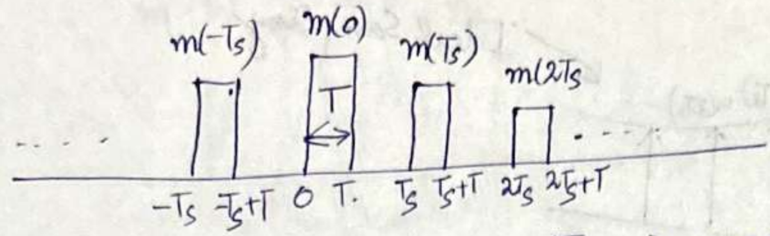
Sampling Interval.

~~Sample~~ ~~Small~~

Employs Flat-top pulses, hence also known as flat-top sampling.

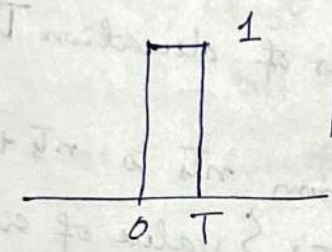
~~At~~

Pulse Amplitude Modulation (PAM)



Amplitude of pulse at nT_s is modulated by signal value at nT_s.

$$m(nT_s) \times p(t - nT_s)$$



pulse $p(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$

At nth sample,

$$m(nT_s) \cdot p(t - nT_s)$$

$$\boxed{m_p(t) = \sum_{n=-\infty}^{\infty} \underbrace{m(nT_s) \cdot p(t - nT_s)}}$$

Shifting every pulse to nT_s

& scaling by $m(nT_s)$

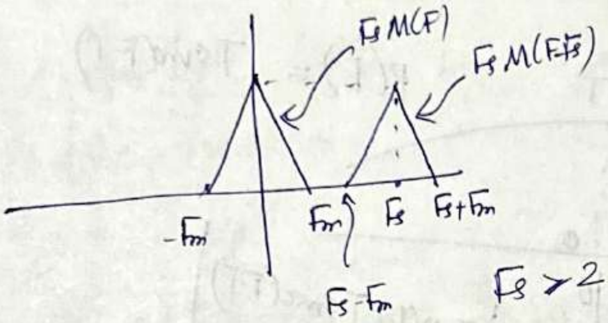
$$\sum_{n=-\infty}^{\infty} m(nT_s)\,\delta(t-nT_s) * p(t) = \sum_{n=-\infty}^{\infty} m(nT_s)\, p(t-nT_s)$$

So, $m_p(t) = m_\delta(t) * p(t)$

↓

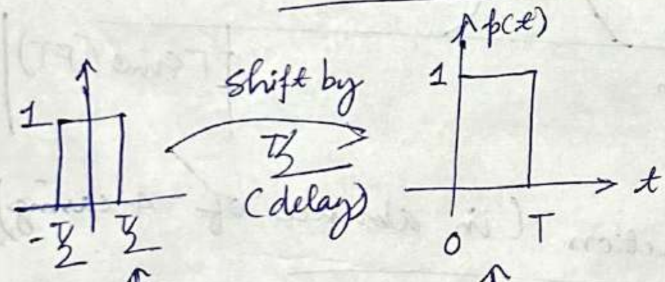
$$M_p(F) = M_\delta(F)\cdot P(F)$$

$$M_\delta(F) = F_s \sum_{k=-\infty}^{\infty} \delta(F-kF_s)$$



$F_s > 2F_m$ to avoid aliasing distortion.

$P(F)$: F.T. of $p(t)$



⇕ $\qquad$ ⇕

$T\,\mathrm{sinc}(F\cancel{T})$

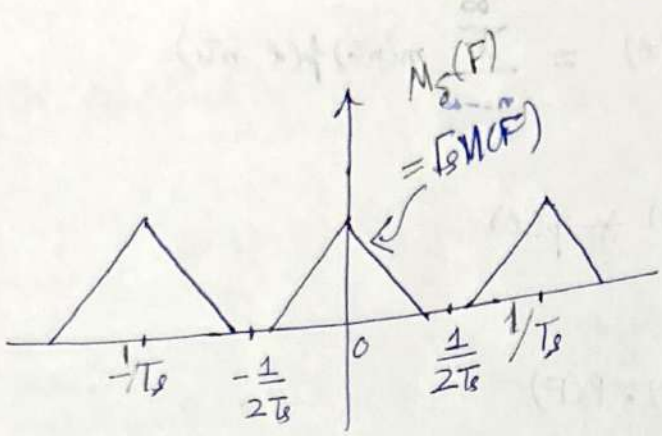
$$e^{-j2\pi F\frac{T}{2}} \times T\,\mathrm{sinc}(FT)$$

$$\boxed{P(F) = T e^{-j\pi FT}\,\mathrm{sinc}(FT)}$$

$$|P(F)| = T\,\mathrm{sinc}(FT)$$

↪ Zeros in frequency Domain are at $F = \frac{k}{T}$ & $k \neq 0$

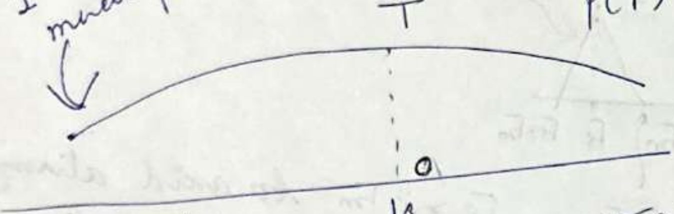
$\cancel{\pi} FT = k\pi \Rightarrow F = \frac{k}{T}, k \neq 0$



$T << T_s \Rightarrow \frac{1}{T} >> \frac{1}{T_s}$

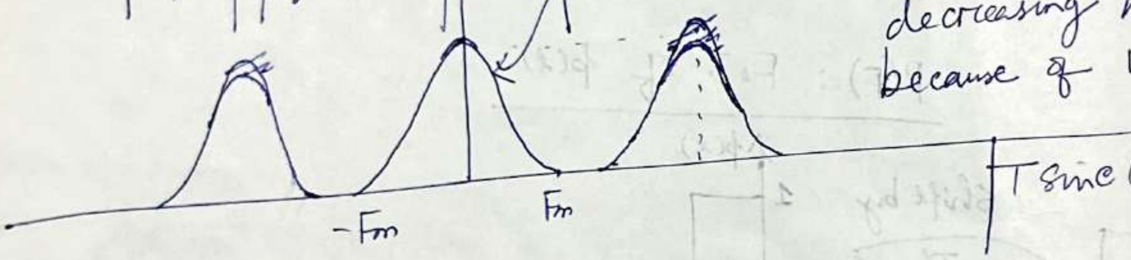
$(1/T)$
1st zero lies much further of $\frac{1}{2T_s}$

$P(F) = T \operatorname{sinc}(FT)$



$|M_p(F)| = |M_\delta(F) . P(F)|$

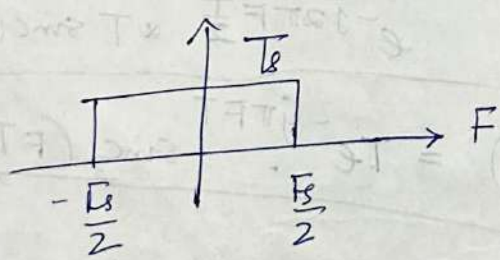
$|F_s M(F) T \operatorname{sinc}(FT)|$



Maximum amplitude is decreasing with frequency because of windowing by $|T \operatorname{sinc}(FT)|$

For reconstruction (in absence of Aliasing)

Step-1



cut off freq $= \frac{F_s}{2}$

Amplitude $= T_s$.

$$M_\delta'(F) = \sum_{k=-\infty}^{\infty} F_s M(F-kF_s)$$

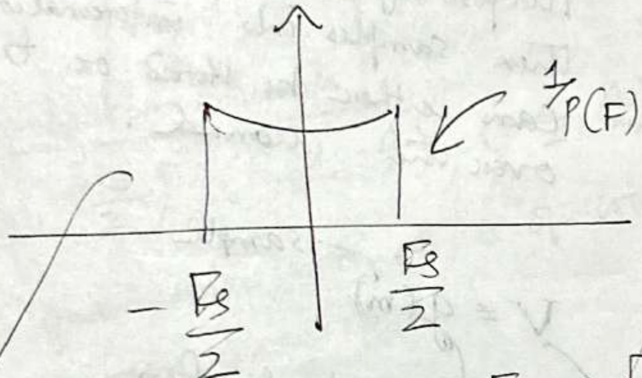
$$P(F) = T\,\mathrm{sinc}(FT)\, e^{-j\pi FT}$$

$$M_p(F) = M_\delta(F)\cdot P(F)$$

$$= \left(\sum_{k=-\infty}^{\infty} F_s M(F-kF_s)\right)\cdot e^{-j\pi FT}\cdot T\,\mathrm{sinc}(FT)$$

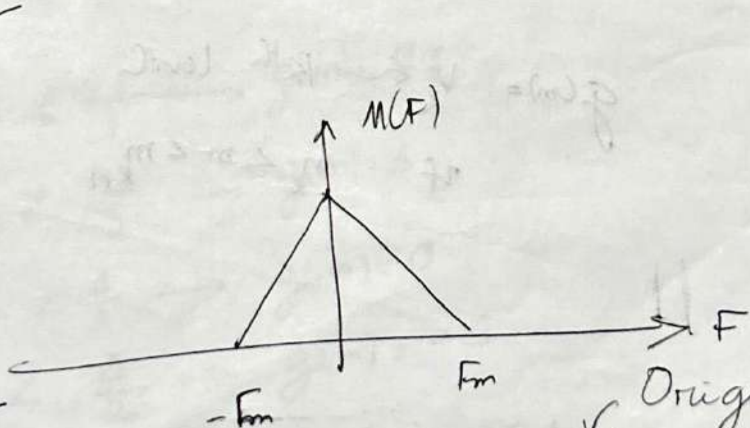
Step 2

Reverse distortion caused by windowing with the sinc function.



Equalizer or Reconstruction Filter

$$\begin{cases} \dfrac{1}{P(F)}, & -\dfrac{F_s}{2} \le F \le \dfrac{F_s}{2} \\ 0, & \text{otherwise} \end{cases}$$

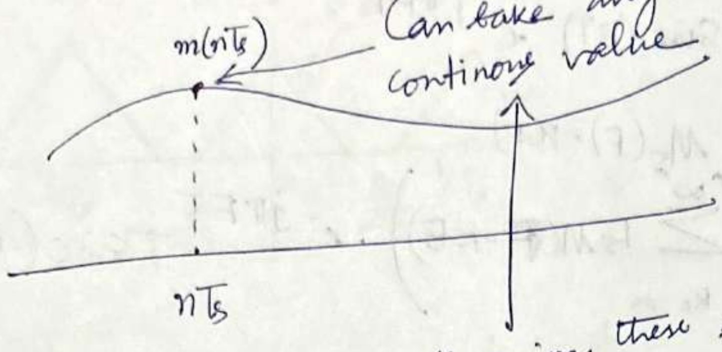


PAM Signal → $m_p(t)$ → [LPF] → [Equalizer] → $m(t)$ ← Original Message Signal

LPF: cut off $= \frac{F_s}{2}$; Equalizer: $\frac{1}{P(F)}$

$\underbrace{\text{LPF} + \text{Equalizer}}$ → Combined into a single Filter

QUANTIZATION



Mapping these samples to a discrete set of values is termed as Quantization

Purpose of Quantization is to convert these samples into information bits which can either be stored or transmitted over the channel.

$$V = g(m)$$

sample

Quantization fun.

belongs to a discrete set of values also termed as Quantization Levels

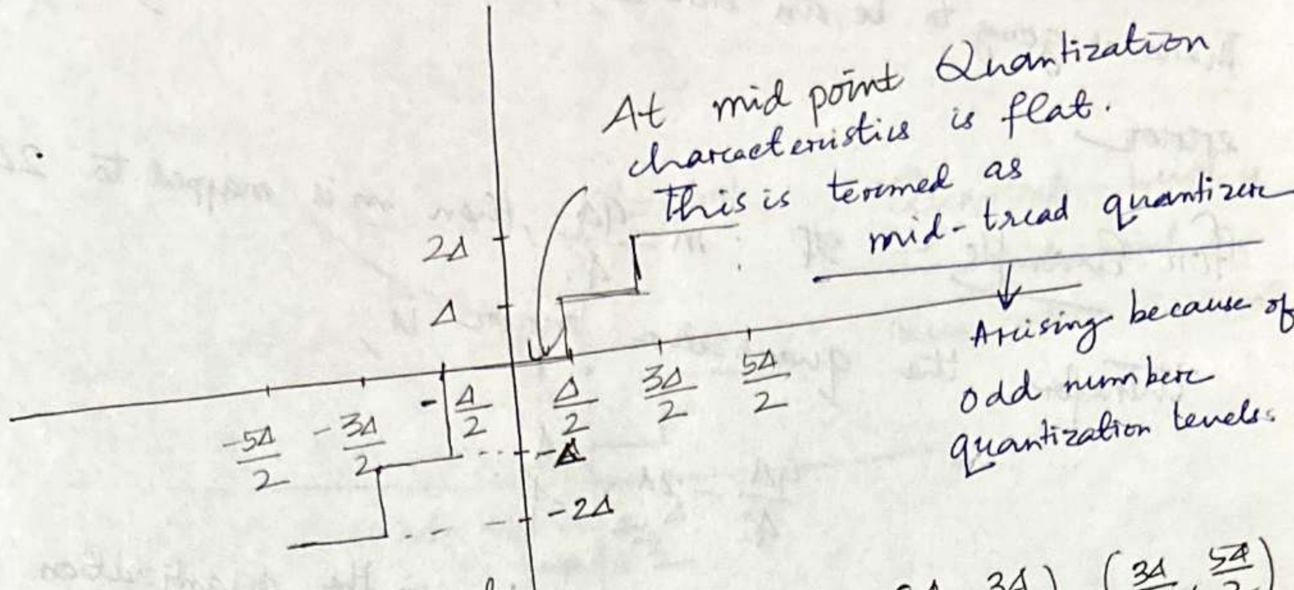
$$g(m) = V_K \leftarrow K^{th} \text{ level}$$

$$\text{if} \quad m_k \leq m \leq m_{k+1}$$

Uniform Quantizer

Quantization Levels are uniformly spaced

Ex:- Let us consider a 5 level Quantizer



5 Quantization intervals

$\left(-\frac{5\Delta}{2}, \frac{-3\Delta}{2}\right), \left(-\frac{3\Delta}{2}, -\frac{\Delta}{2}\right), \left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right), \left(\frac{\Delta}{2}, \frac{3\Delta}{2}\right), \left(\frac{3\Delta}{2}, \frac{5\Delta}{2}\right)$

Each quantization interval is of width Δ. Therefore it is a uniform quantizer.

In each quantization interval, the quantization Level is the mid point of the interval.

$$\frac{3\Delta}{2} \leq m < \frac{5\Delta}{2} \longrightarrow g(m) = 2\Delta$$

$$\frac{\Delta}{2} \leq m < \frac{3\Delta}{2} \longrightarrow g(m) = \Delta$$

$$-\frac{\Delta}{2} \leq m < \frac{\Delta}{2} \longrightarrow g(m) = 0$$

$$-\frac{3\Delta}{2} \leq m < -\frac{\Delta}{2} \longrightarrow g(m) = -\Delta$$

$$\frac{-5\Delta}{2} \leq m < -\frac{3\Delta}{2} \longrightarrow g(m) = -2\Delta$$

For example,

All values in $\left(\frac{3\Delta}{2}, \frac{5\Delta}{2}\right)$ are mapped to 2Δ.

Therefore, this is a many to one mapping.

There is going to be an error, this is known as quantization error.

for Example: If $m = \frac{9\Delta}{4}$, then m is mapped to 2Δ

therefore, the quantization error is,

$$\frac{9\Delta}{4} - 2\Delta = \frac{\Delta}{4}$$

So, depending on where it lies in the quantization interval, the quantization error lies between

$$-\frac{\Delta}{2} \text{ to } \frac{\Delta}{2}$$

Minimum sample value $= \frac{-5\Delta}{2}$

the maximum sample value $= +\frac{5\Delta}{2}$

Quantization Level = ~~Midpoint of the interval.~~

~~When~~

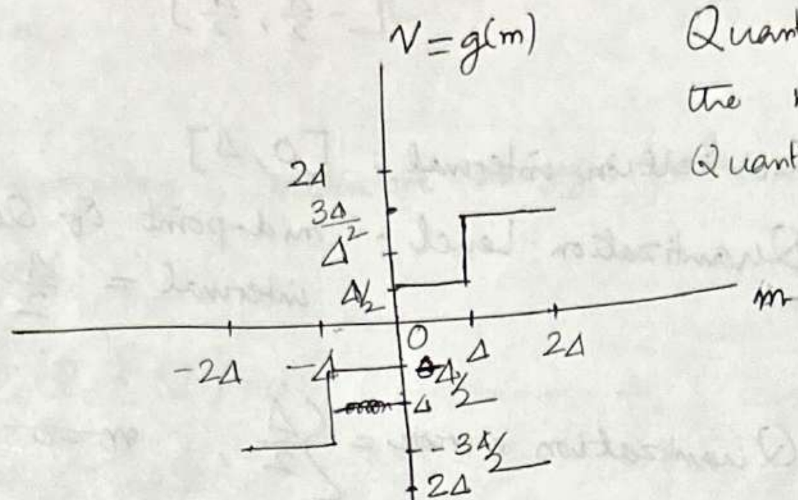
Odd number of Quantization levels ⇒ Mid tread quantizer

Smaller the quantization level, smaller will be the quantization error.

Uniform Quantization

Even number of quantization Levels.

Conside 4 quantization levels, Quantization interval of width Δ.



Quantization Level is the mid-point of Quantization interval.

With an even number of quantization levels the quantizer characteristics rises from $-\frac{\Delta}{2}$ to $\frac{\Delta}{2}$ at $m=0$. This is termed as a mid-rise quantizer

Δ : Quantization interval (or) step size

$m \in [-m_{max}, m_{max}]$ → Peak +ve sample value

Total quantization interval or Dynamic signal range $= 2m_{max}$.

No of quantization levels $= \boxed{L = \frac{2m_{max}}{\Delta}}$

$$\boxed{\Delta = \frac{2m_{max}}{L}}$$

Quantization error:

Quantization is a many-to-one mapping in which all sample values in a particular interval are mapped to a quantization level. $\Rightarrow$ Quantization error.

In a uniform quantizer, the quantization error lies in

$$\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$$

Quantization interval : $[0, \Delta]$

Quantization Level : mid-point of Quantization interval $= \frac{\Delta}{2}$

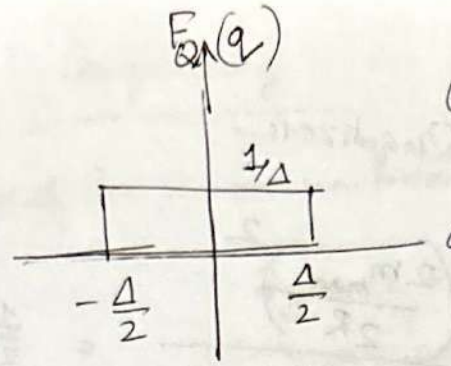
$\Rightarrow$ Quantization error $= \begin{cases} \frac{\Delta}{2}, & \cancel{m=0} \\ -\frac{\Delta}{2}, & \cancel{m=1} \end{cases}$

$\cancel{\frac{g(m)-m}{g}}$

$$\boxed{q = Q(m) - m}$$

Quantized Sample Value ($Q(m)$)

True Sample Value (m)



Quantization error is uniformly distributed in the interval: $\left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right]$

Prob. density fn of Quantization error

$$F_Q(q) = \begin{cases} \frac{1}{\Delta}, & |q| \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

By symmetry, mean or average value of quantization error,

$$E\{Q\} = 0$$

$$\Rightarrow \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} F_Q(q) \cdot q \, dq = 0$$

$E\{Q^2\}$ → Variance or power of Quantization error
Also termed as quantization noise

$$= \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} q^2 F_Q(q) \, dq = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \frac{1}{\Delta} q^2 \, dq = \frac{1}{\Delta}\left[\frac{q^3}{3}\right]_{-\Delta/2}^{\Delta/2}$$

$$= \frac{1}{\Delta} \times \frac{1}{3}\left[\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right] = \frac{1}{3\Delta} \times \frac{2\Delta^3}{8} = \frac{\Delta^2}{12}$$

$$\boxed{\text{Quantization noise power} = \frac{\Delta^2}{12}}$$

$$\Delta = \frac{2m_{max}}{L} = \frac{2m_{max}}{2^R}$$

$R = \log_2 L$ → no of bits required to represent L levels in binary

R : Resolution of Quantizer

$$\sigma_q^2 = \frac{\Delta^2}{12} = \frac{\left(\frac{2 m_{max}}{2^R}\right)^2}{12} = \frac{1}{3} \frac{m_{max}^2}{2^{2R}}$$

Quantization noise variance in dB.

↓

$$10 \log_{10} \sigma_q^2 = -10 \log_{10} 3 + 20 \log_{10} m_{max} - 20 R \log_{10} 2$$

$$\boxed{10 \log_{10} \sigma_q^2 = -10 \log_{10} 3 + 20 \log_{10} m_{max} - 6R}$$

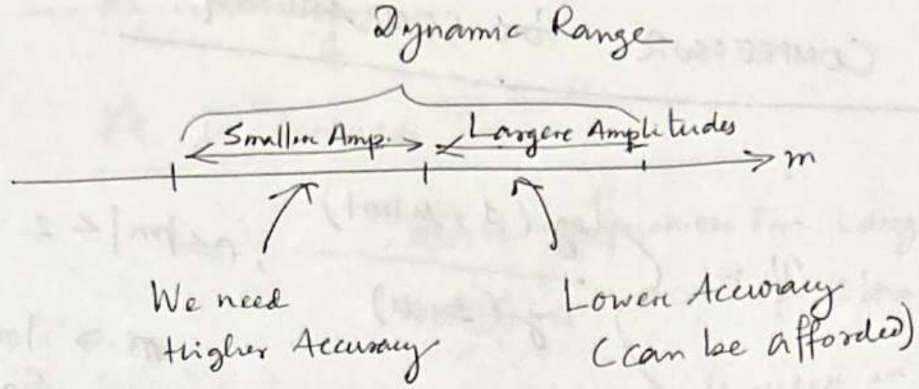
dB noise power decreases by 6dB for each additional bit

SNR : Signal to Quantization noise power ratio

improves by 6dB for each additional bit.

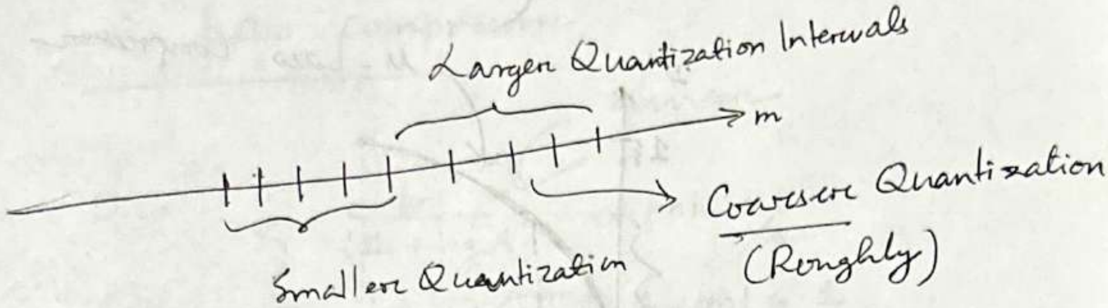
Companding

used for non-uniform quantization.



⇒ Lower Quantization Error

⇒ Higher Quantization Error



Summary: ① At lower amplitudes, we need lower Quantization error ⇒ Smaller Quantization Intervals
⇒ Higher Accuracy of Reconstruction.

② At larger amplitudes, we can tolerate Large Quantization error ⇒ Larger Quantization Interval
⇒ Lower Accuracy of Reconstruction.

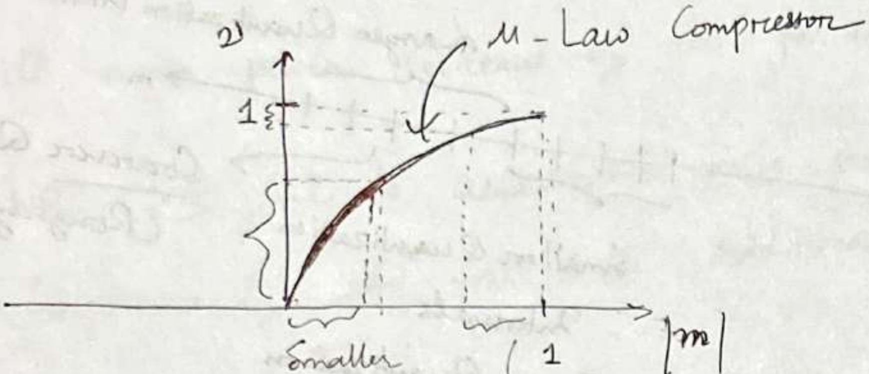
※ Width of Quantization intervals are not same ⇒ Non-uniform Quantization.

COMPANDING Achieves NON-UNIFORM Quantization.

μ- LAW COMPRESSOR for COMPANDING

$$v = \begin{cases} \dfrac{\log(1+\mu|m|)}{\log(1+\mu)}, & 0 \le |m| \le 1 \end{cases}$$

$m \rightarrow$ Normalized Sample value



Smaller Amplitudes

→ Expanded

⇒ Finely Quantized

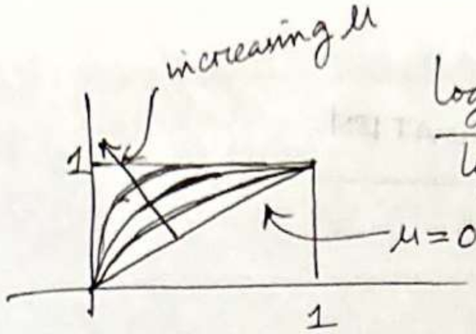
Large Amplitudes

→ Coarsely Quantized

$$\lim_{\mu \to 0} \frac{\log(1+\mu|m|)}{\log(1+\mu)} \quad \left\{ \lim_{x \to 0} \log(1+x) \simeq x \right\}$$

$$= \frac{\mu|m|}{\mu} = |m|$$

$$\underbrace{\phi(|m|) = |m|}_{\text{Linear Chr.}}$$



$$\frac{\log(1+\mu|m|)}{\log(1+\mu)}$$

As μ increases

more compression of Larger amplitudes
more expansion of smaller "

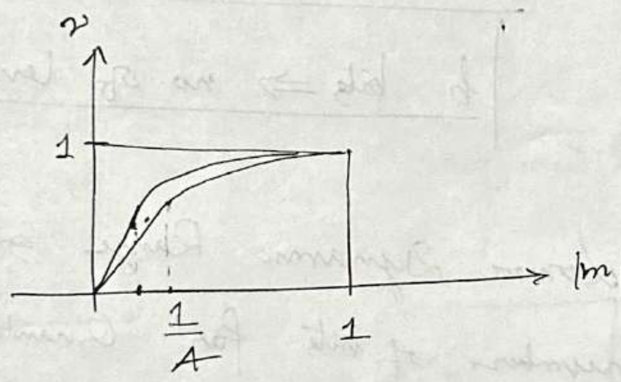
μ- Law compression is used in North American Standards.

A- Law compressor

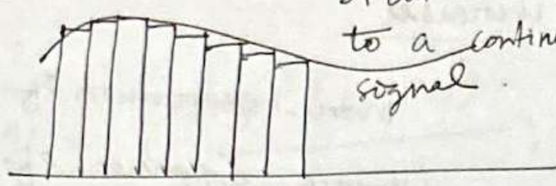
Linear

$$v = \begin{cases} \dfrac{A|m|}{1+\log A}, & 0 \le |m| \le \frac{1}{A} \\ \dfrac{1+\log A|m|}{1+\log A} & \frac{1}{A} \le |m| \le 1 \end{cases}$$

Logarithmic or Concave



DELTA MODULATION



Staircase approximation to a continuous message signal.

Instead of Quantizating samples, quantize the difference between successive samples.

Typically, we have smooth continuous waveform

⇒ Difference of signal samples is small

⇒ Dynamic range of difference between successive samples is very small.

⇒ Finer Resolution {Quantization interval small} & Less Quantization Error

⇒ $\boxed{b \text{ bits} \Rightarrow \text{no. of levels} = 2^b}$

⇒ Lower Dynamic Range ⇒ We need lower number of bits for Quantization.

⇒ Bit Rate can be reduced.

In Delta Modulation, resolution of Quantizer is drastically reduced

In fact we employ only a single bit for quantization.

$m(n) \leftarrow$ Sample at time instant n.

$m_q(n-1) \leftarrow$ Approximation to $m(n)$

Error, $\boxed{e(n) = m(n) - m_q(n-1)}$ — ①

Quantize $e(n)$:

$$\boxed{e_q(n) = \Delta \, sgn(e(n))} \quad ②$$

Quantized Error

$$sgn(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases}$$

If $e(n) > 0$, $e_q(n) = +\Delta$

If $e(n) < 0$, $e_q(n) = -\Delta$

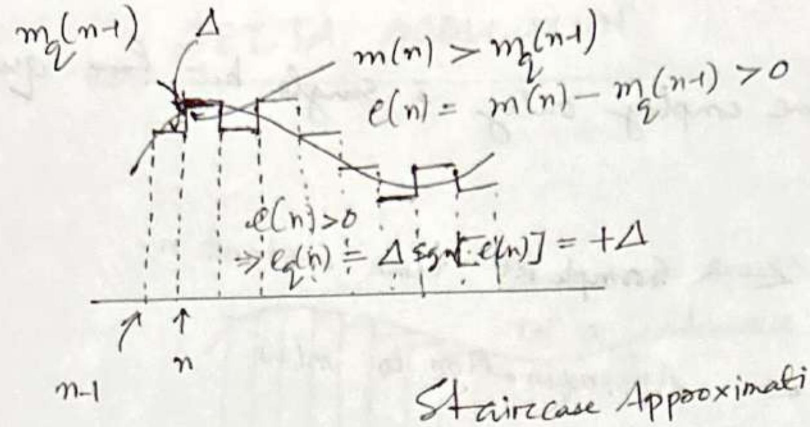
$e_q(n)$ can take only 2 values $\{-\Delta$ or $+\Delta\}$

Hence, only 2 levels

Therefore only 1 bit is required for quantization.

$$\boxed{m_q(n) = m_q(n-1) + e_q(n)} \quad ③$$

Reconstruct Prediction or approximation for next time instant



Staircase Approximation of Signal

$$e_q(n) = \begin{cases} +\Delta \\ -\Delta \end{cases}$$

$\downarrow$

Encodized using 1 bit

$\Delta \quad -\Delta \quad -\Delta \quad \Delta \quad \Delta \quad -\Delta$

$1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0$

Encoded Signal

Can be used for signal compression

→ Speech Compression

→ Image Compression

Delta Modulation

Differential Modulation

Encodes the difference of signal samples.

Each sample is encoded using a single bit only

Hence it is a very efficient modulation.

↓ Results in a low bit rate

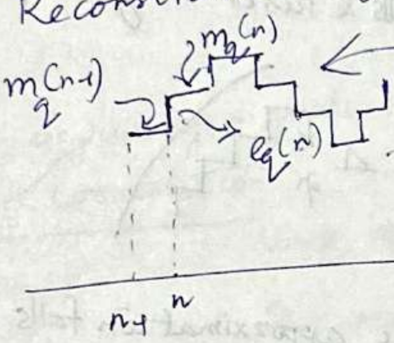
Can also be used for compression.

$m_q(n-1)$: Quantized approximation of signal at time instant n. (Also, reconstruction at $n-1$)

$e_q(n)$: Quantized error at time n. Either $+\Delta$ or $-\Delta$

$$\boxed{m_q(n) = m_q(n-1) + e_q(n)}$$ ← Eqn for signal reconstruction at n.

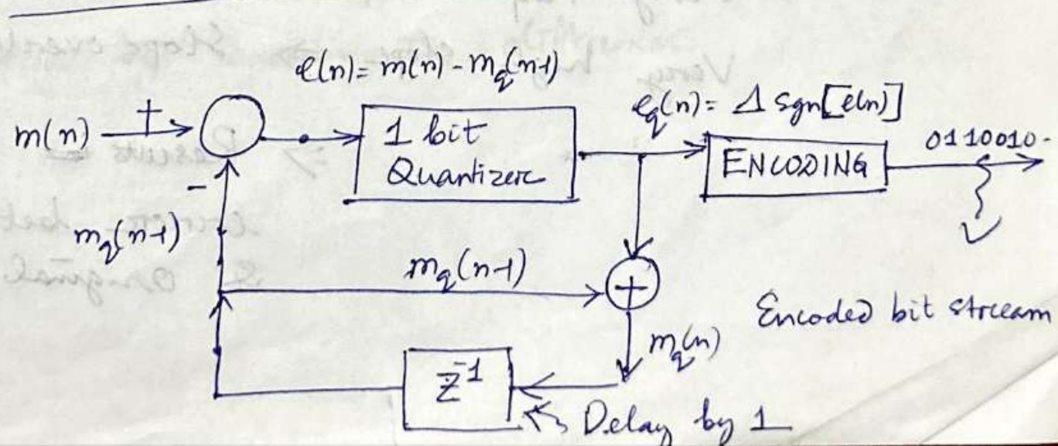
↓ Reconstructed signal at time n



Staircase approximation of Original signal

Pass through LPF → After passing through an LPF, Smothened version of signal is obtained.

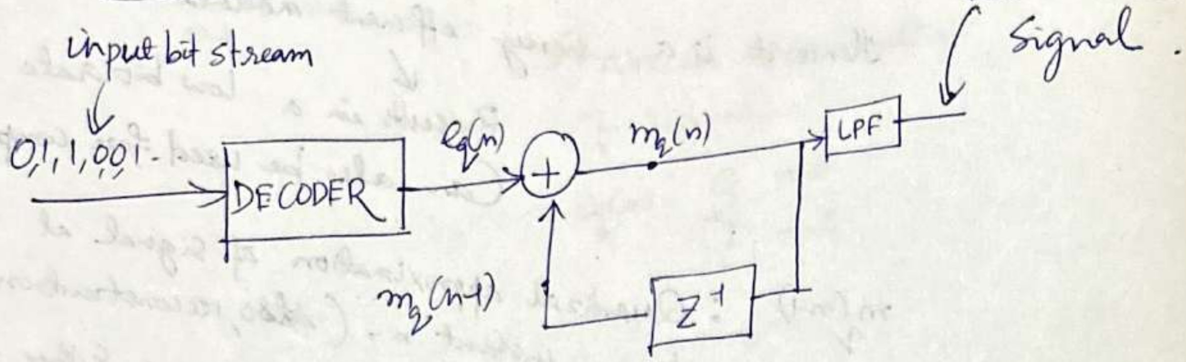
Schematic Diagram of Delta ~~Modulation~~ Modulation.



Encoded bit stream can be digitally modulated &

transmitted over comm. channel.

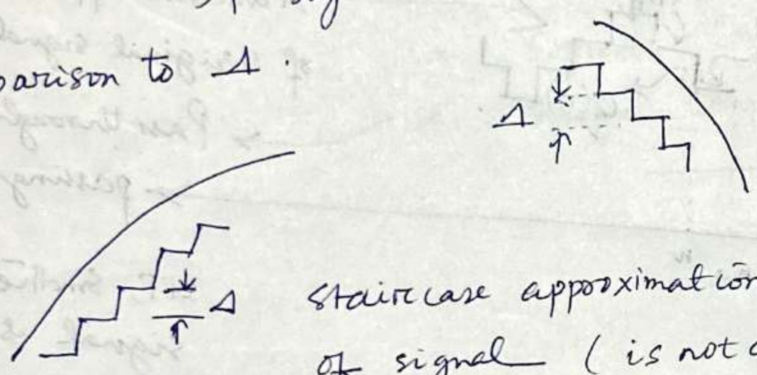
Schematic for Reconstruction:



Distortion in Delta modulation:

Too Small size Δ (stepsize)

If signal falls & rises very rapidly in comparison to Δ.

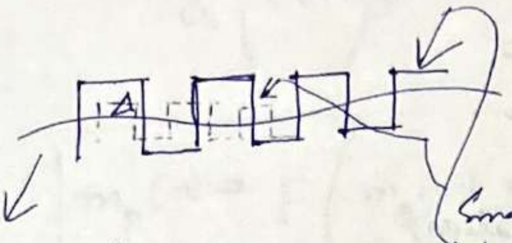


staircase approximation falls short of signal (is not able to catch up the signal.

Very rapid rise or fall in the signal $\Rightarrow$
Very high slope $\Rightarrow$ Slope overload Distortion.

$\Rightarrow$ Results in a high error between reconstructed & Original signal.

Too Large Step Size :-



Signal is relatively Flat.

Smaller $\Delta \Rightarrow$ Smaller Error
Larger $\Delta \Rightarrow$ Larger Error

Granular noise $\rightarrow$ Arising from large value of step size Δ. When the signal is relatively flat.

Different Pulse Modulation Schemes

DPCM (Differential Pulse Coded Modulation) (DPCM)

Naturally occuring signals :- video and audio signals are very high correlation $\Rightarrow$ High Redundancy

Bit rate can be significantly reduced by reducing the redundancy of the signal.

DPCM acheives this by predicting the signal m(n) & followed by encoding the difference.

$$m_p(n) = F\Big(\overbrace{m_q(n-1), m_q(n-2), \cdots, m_q(n-L)}^{L \text{ past quantized samples}}\Big)$$

Prediction of signal at n

For Delta modulation,

$$\boxed{m_p(n) = m_q(n-1)}$$

Prediction function for Delta Modulation

$$\underbrace{m_q(n-1), m_q(n-2), \cdots \quad m_q(n-L)}_{L\text{-past quantized samples.}}$$

$F(\cdot) \rightarrow$ Prediction function.

$\hookrightarrow$ It has to be appropriately designed

- Better prediction $\Rightarrow$ Higher efficiency

DPCM

$$\boxed{e(n) = m(n) - m_p(n)} \quad -② $$

Prediction error

$$\boxed{e_q(n) = Q\big(e(n)\big)} \quad -③$$

Quantized prediction error

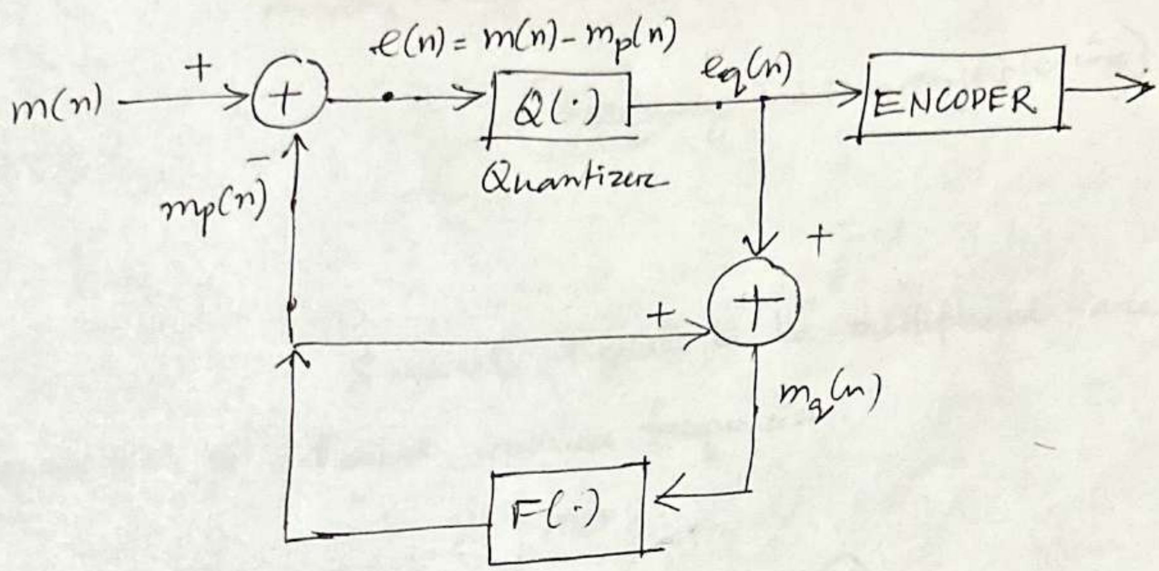
If Prediction is good $\Rightarrow e(n)$ is lower $\Rightarrow$ [quan]tization is efficient.

$$\boxed{m_q(n) = m_p(n) + e_q(n).} \rightarrow \text{Reconstruction.} \quad (4)$$

$Q(e(n)) \rightarrow$ can have arbitrary no. of bits.

$$\boxed{m_p(n) = F\left(m_q(n-1), m_q(n-2), \ldots, m_q(n-L)\right)} \quad -(1)$$

Schematic Diagram

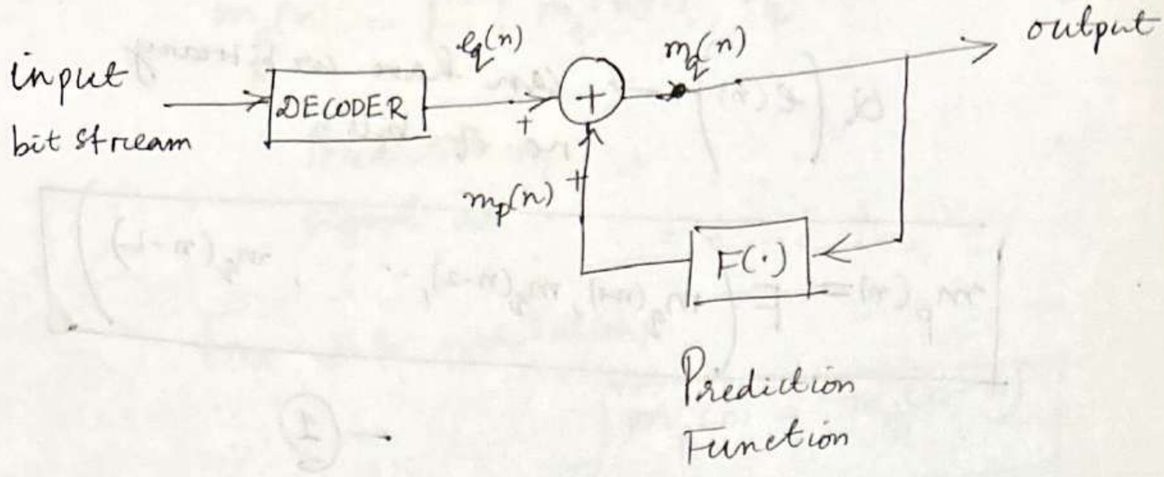


Prediction Function. ← For Delta Modulation simply z^{-1}

DPCM Reconstruction

$$m_p(n) = F\left(m_q(n-1), m_q(n-2) \cdots m_q(n-L)\right).$$

$$m_q(n) = m_p(n) + e_q(n)$$


input
bit stream
DECODER
$e_q(n)$
+
+
$m_q(n)$
$m_p(n)$
output
$F(\cdot)$
Prediction
Function

Frequency Division Multiplexing

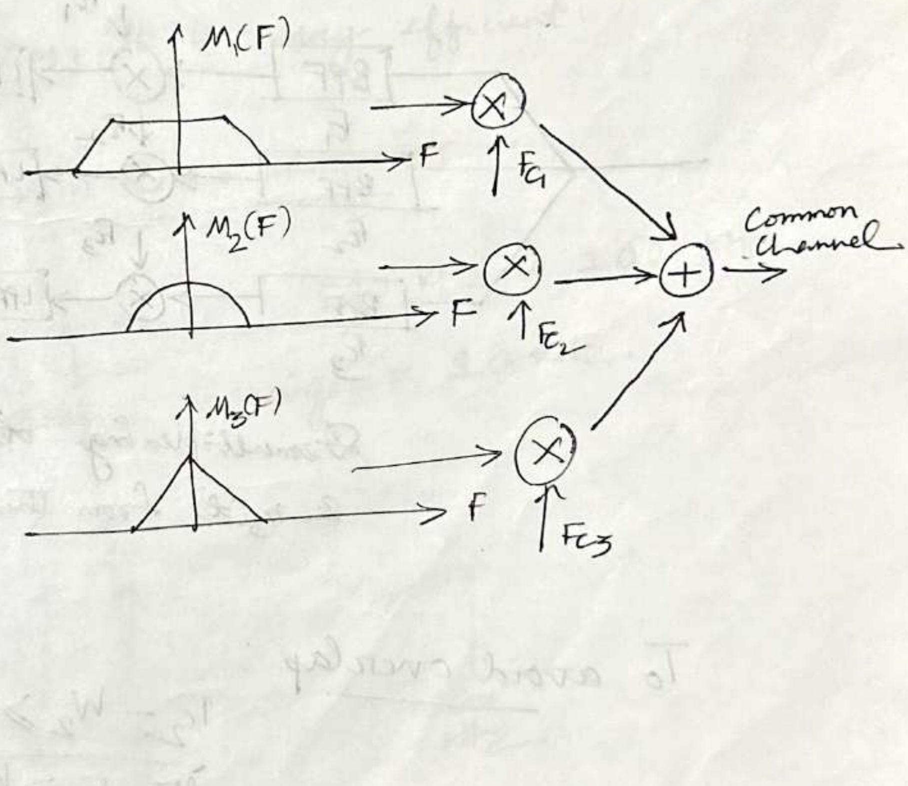
Multiplexing :- Multiple signals are combined into a single composite signal for transmission over a channel

Same channel is used for the transmission of multiple signals.

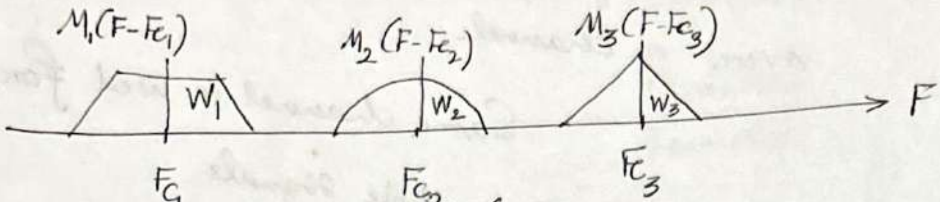
multiplexing is done in the frequency domain
↓
(Frequency Division Multiplexing)

How?

Several signals to be multiplexed are translated to different carrier frequencies.



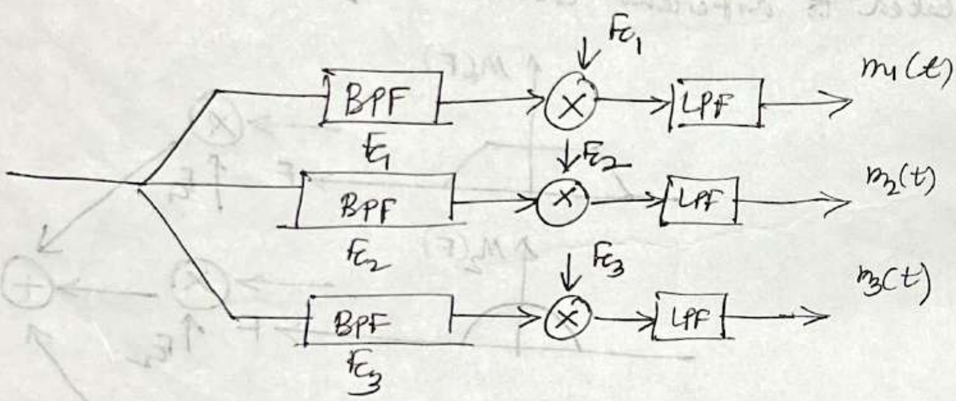
Spectrum of Composite Signal



Transmitted over common channel.

$m_1(t)$, $m_2(t)$, $m_3(t)$ are multiplexed in Frequency domain for transmission over channel.

→ (FDM)



Demultiplexing to extract $m_1(t)$, $m_2(t)$ & $m_3(t)$ from the composite signals.

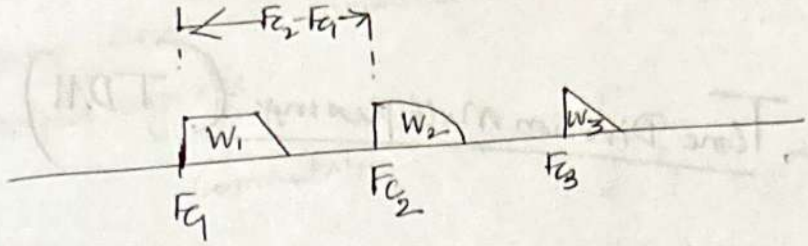
To avoid overlap:

$$F_{c_2} - W_2 \geq F_{c_1} + W_1$$

$$\Rightarrow \boxed{F_{c_2} - F_{c_1} \geq W_1 + W_2}$$

Carrier Frequency seperation $\geq$ Sum of bandwidth

For SSB Modulation



For no overlap, $F_{c_2} \geq F_{c_1} + W_1$

$$\Rightarrow \boxed{F_{c_2} - F_{c_1} \geq W_1}$$

$$\boxed{F_{c_3} - F_{c_2} \geq W_2}$$

Spacing $\geq$ Bandwidth

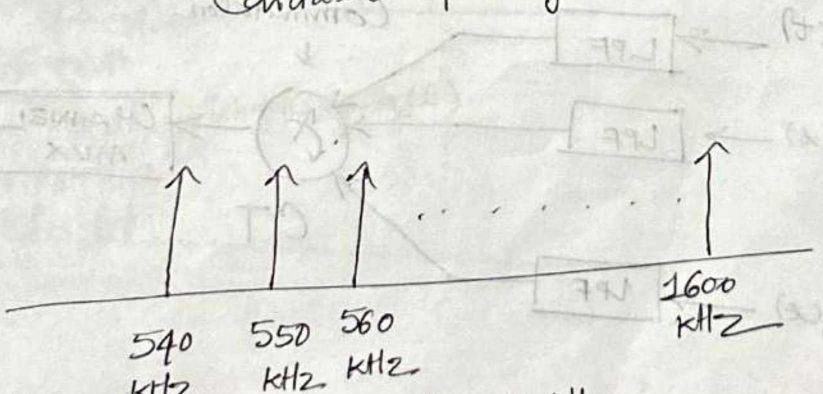
$\Rightarrow$ Total Bandwidth required for FDM + SSB $<<$ FDM + DSB.

$\Rightarrow$ FDM + SSB is very efficient.

Example: AM Radio BROADCAST

Carrier in 540 kHz to 1600 KHz

Carrier spacing = 10 KHz



Carrier spacing is only 10kHz

$W \simeq 15 \text{ kHz} > \text{Carrier Spacing}$

Adjascent Carriers are not used in same geographic location in AM radio broadcast.

Time Division Multiplexing (TDM)

Application of Sampling.

Several signals are multiplexed in the time domain to form a composite signal for transmission over a channel.

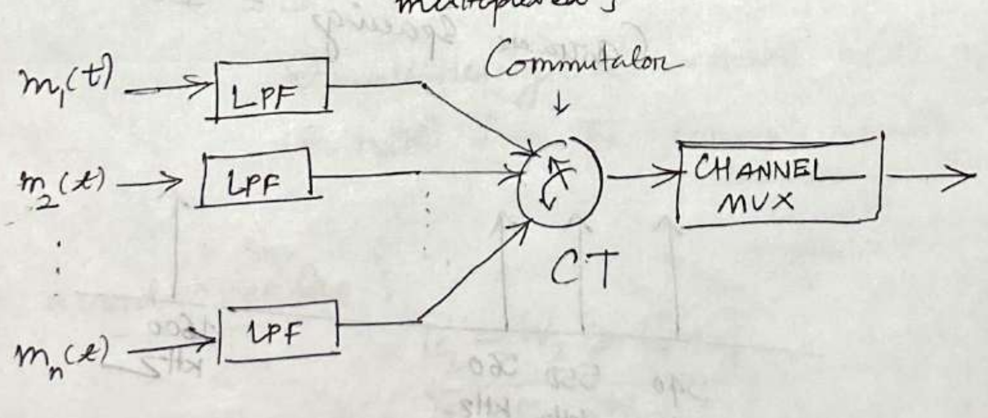
TDM Operation

→ Each signal is Low pass filtered to first remove the undesired frequencies.

→ Then signals are applied to a commutator which alternatively samples the signals to be multiplexed.

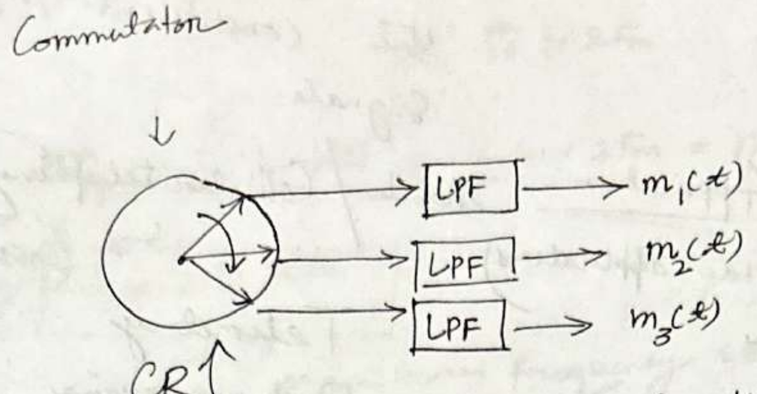
→ This commutator is implemented using high speed electronic switching circuit.

{ $m_1(t), m_2(t), \dots m_n(t)$ are n signals to be multiplexed }



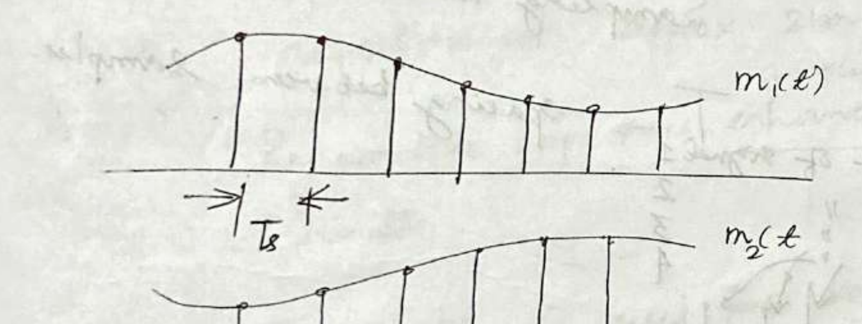
Samples are multiplexed to form a composite signal which is transmitted over a channel.

At the receiver the composite signal is demultiplexed using another commutator followed by low pass filtering of output signals.



Demultiplexes samples from the composite signal.

CT & CR have to be synchronised for error free operation.



Duration between samples in TDM signal is,

$$\boxed{T = \frac{T_s}{n}}$$

n : Number of samples to be multiplexed.

T_s : Sampling interval for each of the constituent signals of the TDM signal.

Application : Mobile / Cellular telephony → (T1 System)
(wide applications) ↳ (GSM, → 2G Wireless Standard)

Telemetry

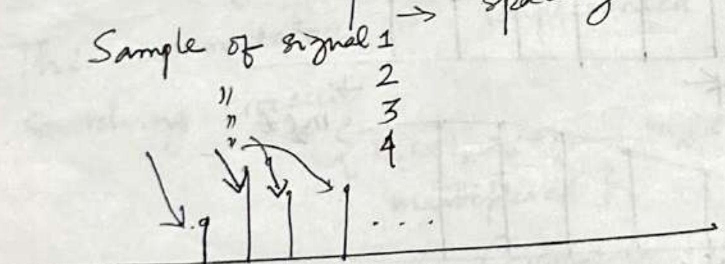
Data processing

Bandwidth required for TDM

$n \rightarrow$ no of signals to be multiplexed.

Sampling interval for each signal $= T_s$

$T \rightarrow$ spacing between samples of TDM



Each signal contributes 1 sample in T_s

$\Rightarrow$ n signals contribute n samples in T_s

So, $\boxed{T = \frac{T_s}{n}}$

BW required for transmission

$$F_{TDM} = \frac{1}{2T} = \frac{1 \times n}{2 \times T_s} = \frac{n}{2} F_s = \frac{1}{2} n F_s$$

$F_s \rightarrow$ Sampling frequency

From Nyquist criterion, we need

$$F_s \geq 2F_m$$

$$F_{TDM} = \frac{1}{2} n F_s \geq \frac{1}{2} \times n \times 2F_m = nF_m$$

$$\Rightarrow \boxed{F_{TDM} \geq nF_m}$$

F_m : maximum frequency component ~~of~~

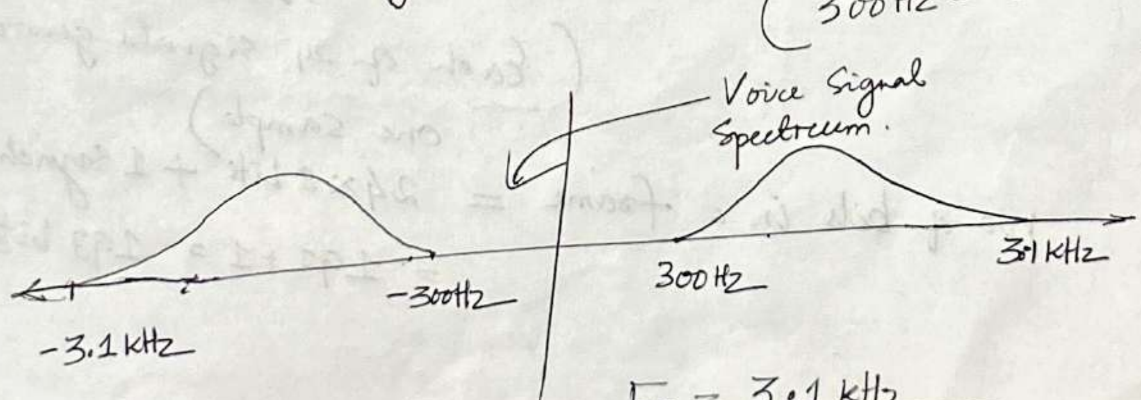
Case study for TDM System

T_1 System (Voice Telephony)

It Comprises of 24 voice channels over seperate pairs of wires with regenerative repeaters at approx. 2km intervals.

Regenerative repeaters enhances (amplify) the signal (TDM) strength.

T_1 system is used for voice communication (300Hz - 3.1 KHz)



$F_m = 3.1$ kHz

From Nyquist criterion to avoid distortion. (Aliasing)

$$F_s \geq 2F_m = 2 \times 3.1\,kHz = 6.2\,kHz$$

$$\Rightarrow \boxed{F_s \geq 6.2\,kHz}$$

$\hookrightarrow$ Nyquist Sampling rate to avoid aliasing

Typical sampled at

$$\boxed{F_s = 8\,kHz}$$

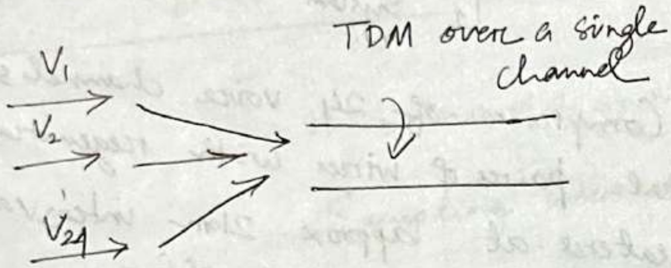
$\downarrow$ 8000 samples/sec

Each sample is quantized using 8 bits. $\{2^8 = 256$ Levels$\}$

$\downarrow$

(μ-Law Quantizer)

8 bits/samples from each of $n = 24$ voice signals.



Sampling rate = 8 kHz

$\Rightarrow$ Sampling duration, $T_s = \frac{1}{8\,Hz} = 125\,\mu s$

$\swarrow$

Frame Duration

(Each of 24 signals generate one sample)

no of bits in a frame $= 24 \times 8$ bits $+ 1$ synch pulse

$= 192 + 1 = 193$ bits

In T_1 system, no. of bits/frame = 193

$\Rightarrow$ Time duration between bits $= \dfrac{125\,\mu s}{193 \text{ bits}}$

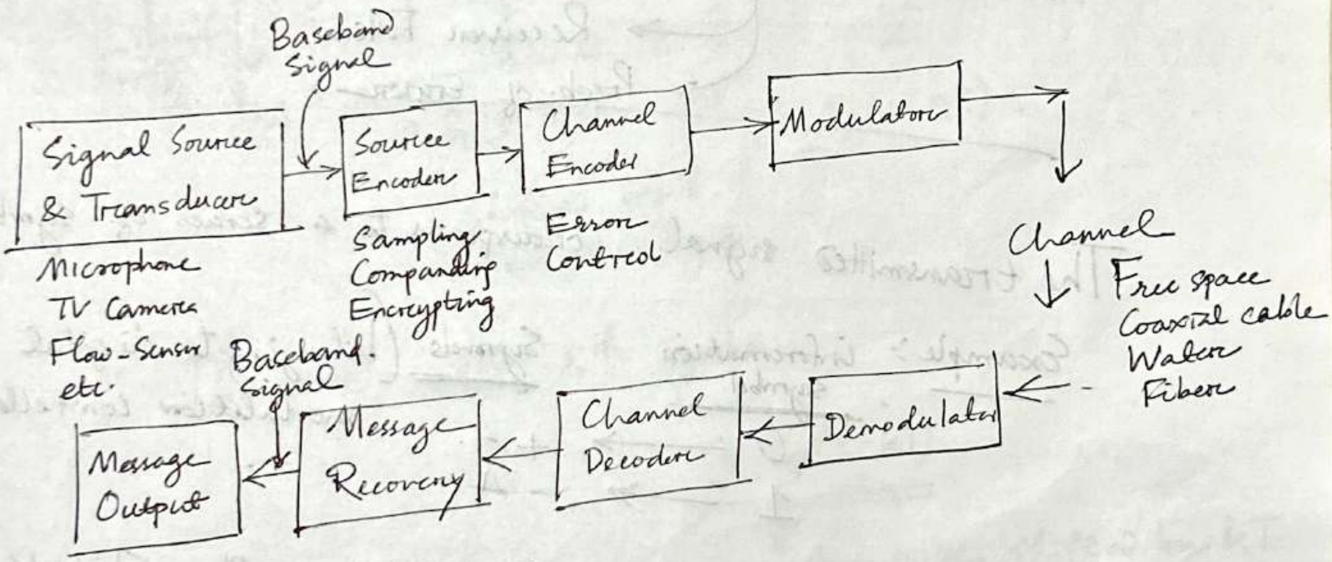
$= 0.647\,\mu s/\text{bit}$

Bit rate of T_1 System is : $\dfrac{193 \text{ bits}}{125\,\mu s} = 1.544$ Mega Bits/sec

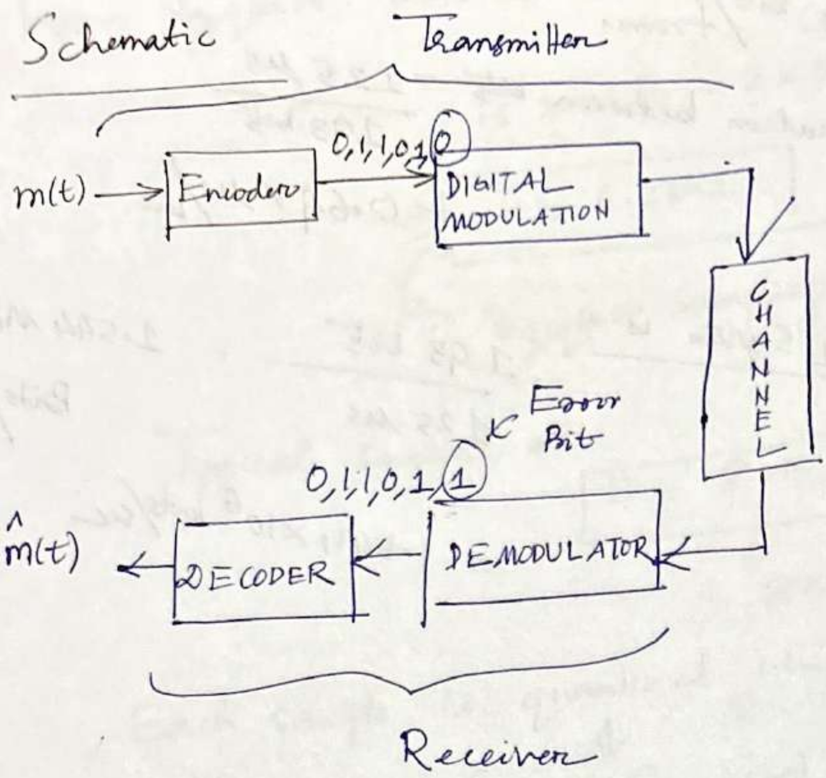
$= 1.544 \times 10^6$ bits/sec

Part-2

DIGITAL COMM. SYSTEM



Schematic



How to design various modules?

How to transmit? ← Transmit Power, Modulation

How to receive?
- → Receiver Filter
- → Prob. of Error

The transmitted signal corresponds to a series of symbols.

Example :- information symbol		Symbols (belonging to digital modulation constellation)
0	→	+A
1	→	−A

BPSK (Binary Phase Shift Keying)

This mapping operation is termed as Digital Modulation.

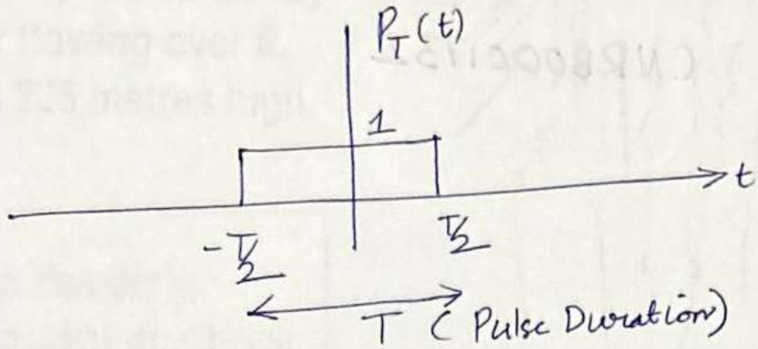
Bit stream

0, 1, 1, 0, 1, 0, 1, 1, 0

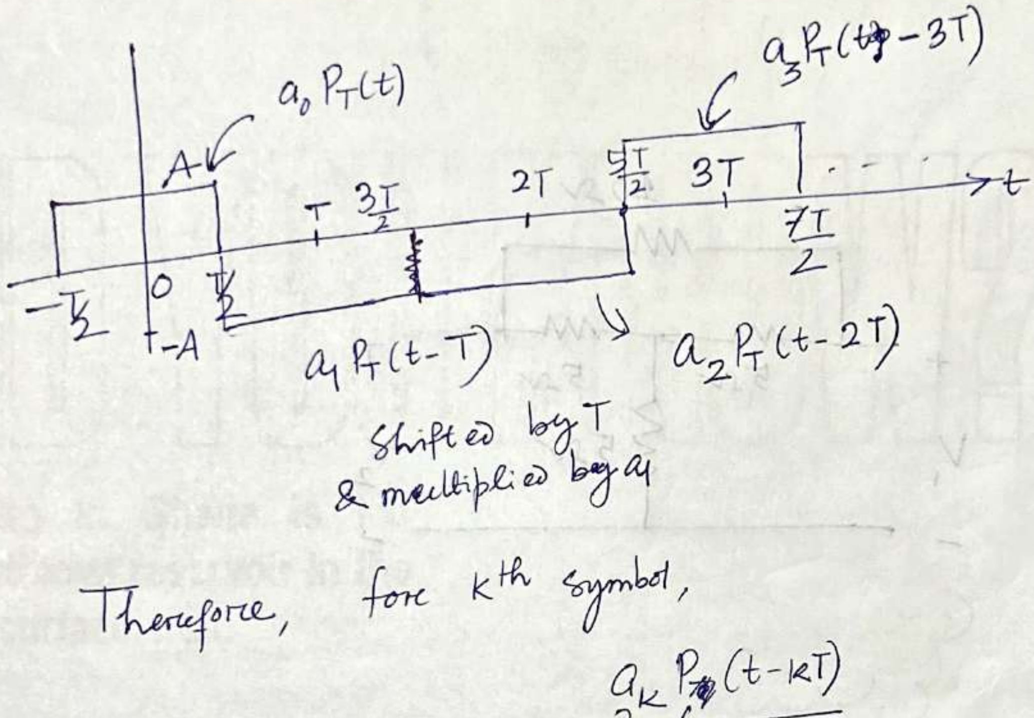
A −A −A A −A A −A −A A

Stream of digitally modulated symbols.

Transmit the digitally modulated symbols using a pulse.



$$P_T(t) = \begin{cases} 1, & |t| \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$



Therefore, for k^{th} symbol,

$$a_k P_T(t-kT)$$

a_k is the k^{th} symbol.

Any pulse shifted by kT.

Net transmitted digital comm. signal is

$$\boxed{x(t) = \sum_{k=} a_k P(t-kT)}$$

Structure of a typical transmitted signal in a digital comm. system.

Spectrum of the transmitted signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{P_T(t-kT)}_{\text{Pulse shifted by } kT}$$

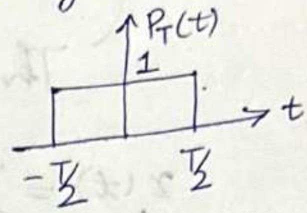
a_k : k^{th} symbol

$$k^{th} \text{ bit} = \underbrace{0 \text{ or } 1}_{\rightarrow \text{ Random}}$$

$$a_k = \underbrace{A \text{ or } -A}_{\rightarrow \text{ Random}}$$

Aim: To characterise spectrum of transmitted signal $x(t)$

$$P_T(t) = \begin{cases} 1, & |t| < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$



$\Updownarrow$ F.T.

$$T \operatorname{sinc} FT = \frac{T \sin(\pi FT)}{\pi FT}$$

Since a_k is random

$$P(a_k) = A = 1/2$$
$$P(a_k) = -A = 1/2$$

$$E\{a_k\} = A \Pr(a_k = A) + (-A)\Pr(a_k = -A)$$
$$= A \times \frac{1}{2} - A \times \frac{1}{2} = 0$$

$$\Rightarrow \boxed{E\{a_k\} = 0}$$

Assume that the symbols a_k are iid (Independent Identically Distributed)

$$E\{a_k a_m\} = E\{a_k\} E\{a_m\} = 0$$

if $k \neq m$

uncorrelated.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k P_T(t-kT)$$

Average value ?

$$E\{x(t)\} = E\left\{\sum_{k=-\infty}^{\infty} a_k P_T(t-kT)\right\} = \sum_{k=-\infty}^{\infty} E\{a_k\} P_T(t-kT)$$

$$\Rightarrow \boxed{E\{x(t)\} = 0}$$

Average value of transmitted signal $x(t)$ is zero.

$$\rightarrow FT\{x(t)\} = X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft} dt$$

$$E\{X(F)\} = E\left\{\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi Ft} dt\right\} = \int_{-\infty}^{\infty} E\{x(t)\} e^{-j2\pi Ft} dt = 0$$

$$\Rightarrow \boxed{E\{X(F)\} = 0}$$

Average value of spectrum transmitted is zero.

This doesnot mean spectrum = 0

$$\underbrace{x(t)}_{\text{random}} = \sum_{k=-\infty}^{\infty} \underbrace{a_k}_{\text{random}} P_T(t-kT)$$

<u>How to measure</u> the spectral content of a random signal?

↳ PSD (Power Spectral Density)

Gives spectral distribution of power of a random signal

<u>To compute PSD</u>

<u>Step-1</u>

Auto-Correlation Function

$$R_{xx}(\tau) = E\{x(t)\, x(t+\tau)\}$$

for a WSS (Wide Sense Stationary Process)

<u>Step-2</u> :-

$$ACF \xleftrightarrow{F.T.} PSD$$

$$E\{x(t)\cdot x(t+\tau)\}$$

$$= E\left\{\left(\sum_{k=-\infty}^{\infty} a_k p_T(t-kT)\right) \times \left(\sum_{m=-\infty}^{\infty} a_m p_T(t+\tau-mT)\right)\right\}$$

$$= E\left\{\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_k a_m p_T(t-kT)\, p_T(t+\tau-mT)\right\}$$

$$= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\{a_k a_m\}\, p_T(t-kT)\, p_T(t+\tau-mT)$$

$$E\{a_k a_m\} = E\{a_k\}\, E\{a_m\} = 0$$

if $k \neq m$

(∵ Symbols are zero mean iid)

$$= \sum_{k=-\infty}^{\infty} E\{a_k^2\}\, p_T(t-kT)\, p_T(t+\tau-kT)$$

(for $k=m$)

$$P(a_k = A) = \frac{1}{2}$$

$$P(a_k = -A) = \frac{1}{2}$$

$$E\{a_k^2\} = \frac{1}{2}A^2 + \frac{1}{2}(-A)^2 = A^2 = \underbrace{P_d}_{\text{Data Symbol Power}}$$

$$= P_d \underbrace{\sum_{k=-\infty}^{\infty} p_T(t-kT)\, p_T(t+\tau-kT)}_{\text{depends on } t}$$

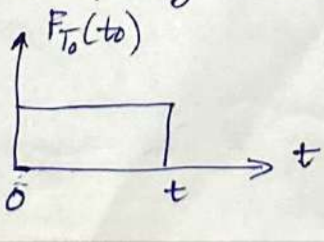
Hence NOT WSS

(Wide Sense Stationary)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k p_T(t-kT-\underbrace{t_0}_{\text{Random Delay}})$$

uniformly distributed in $[0, T]$

uniform prob. Density Function.



$$E\{x(t)\cdot x(t+\tau)\} = P_d \sum_{k=-\infty}^{\infty} \underbrace{E\{P_T(t-kT-t_0)\, P_T(t+\tau-kT-t_0)\}}_{\text{Average/Expected value w.r.t } t_0}$$

$$= P_d \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} F_{T_0}(t_0)\, P_T(t-kT-t_0)\, P_T(t+\tau-kT-t_0)\, dt_0$$

Averaging w.r.t. t_0 (random delay)

$$F_{T_0}(t_0) = \begin{cases} \frac{1}{T}, & 0 < t_0 \le T \\ 0, & \text{otherwise} \end{cases}$$

$$= \frac{P_d}{T} \sum_{k=-\infty}^{\infty} \int_0^T P_T(t-kT-t_0)\, P_T(t+\tau-kT-t_0)\, dt_0$$

$$t_0 + kT = \tilde{t}_0$$

$$dt_0 = d\tilde{t}_0$$

$$= \frac{P_d}{T} \sum_{k=-\infty}^{\infty} \int_{kT}^{(k+1)T} P_T(t-\tilde{t}_0)\, P_T(t+\tau-\tilde{t}_0)\, d\tilde{t}_0$$

Each integral is from kT to $(k+1)T$

$$= \frac{P_d}{T} \cancel{\sum} \int_{-\infty}^{\infty} P_T(t-\tilde{t}_0)\, P_T(t+\tau-\tilde{t}_0)\, d\tilde{t}_0$$

$$-\tilde{t}_0 + t + \tau = t_0'$$

$$\Rightarrow -d\tilde{t}_0 = dt_0'$$

$$= \frac{P_d}{T} \int_{\infty}^{-\infty} P_T(t_0'-\tau)\, P_T(t_0')\, (-dt_0')$$

$$= \underbrace{\frac{P_d}{T} \int_{-\infty}^{\infty} P_T(t_0')\, P_T(t_0'-\tau)\, dt_0'}$$

independent of t

only depends on τ

Hence, $x(t)$ is WSS

$$E\{x(t)\cdot x(t+\tau)\} = R_{xx}(\tau)$$

$$\underbrace{\int_{-\infty}^{\infty} P_T(t_0'-\tau)\cdot P(t_0')\,dt_0'}_{ACF}$$

$R_{P_TP_T}$ ← ACF of pulse P_T

$$\boxed{R_{xx}(\tau) = E\{x(t)x(t+\tau)\} = \frac{P_d}{T} R_{P_TP_T}(\tau)}$$

ACF of signal depends on ACF of pulse $P_T(t)$

F.T.

$$S_{xx}(F) = \frac{P_d}{T} \underbrace{S_{P_TP_T}(F)}$$

Energy Spectral Density of $P_T(t)$

PSD of transmitted signal $x(t)$

$$R_{pp}(\tau) = \int_{-\infty}^{\infty} p(t)\,p(t-\tau)\,dt$$

F.T.

$$S_{pp}(F) = \int_{-\infty}^{\infty} R_{pp}(\tau)\,e^{-j2\pi F\tau}\,d\tau$$

$$\Rightarrow \boxed{S_{pp}(F) = |P(F)|^2}$$

$P(F)$: F.T. of pulse $p(t)$

$$P(F) = \int_{-\infty}^{\infty} p(t)\,e^{-j2\pi Ft}\,dt$$

ESD of $p(t)$

$$R_{xx}(\tau) = \frac{P_d}{T} R_{pp}(\tau)$$

$$S_{xx}(F) = \int_{-\infty}^{\infty} R_{xx}(\tau)\,e^{-j2\pi F\tau}\,d\tau$$

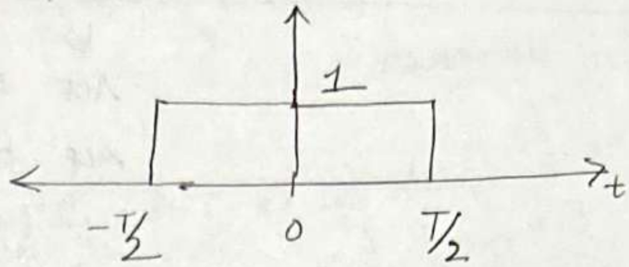
For a WSS process, PSD is given by F.T. of ACF.

$$\boxed{S_{xx}(F) = \frac{P_d}{T} S_{pp}(F) = \frac{P_d}{T} |P(F)|^2}$$

PSD of transmitted signal is proportional to Energy Spectral Density of pulse

Example :

Consider $p(t) = P_T(t)$



$$P_T(F) = T \,\text{sinc}(FT) = T \frac{\sin(\pi F T)}{(\pi F T)}$$

$$|P_T(F)|^2 = T^2 \,\text{sinc}^2(FT)$$

$$PSD = \frac{P_d}{T} \times T^2 \,\text{sinc}^2(FT) = P_d T \,\text{sinc}^2(FT)$$

$$\Rightarrow \boxed{S_{xx}(F) = P_d T \,\text{sinc}^2(FT)}$$ PSD of transmitted signal

DIGITAL COMMUNICATION CHANNEL

Channel :- Medium through which signal travels from transmitter to receiver in a communication system.

Example :- Telephone lines, Coaxial cables, Wireless Channel.

Ex :- AM, FM, Cellular, WiFi, Bluetooth.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, p(t-kT)$$

$$x(t) \longrightarrow \boxed{\text{CHANNEL}} \longrightarrow y(t)$$

Transmitted Signal

Received Signal

Simple Channel model :- (AWGN) channel
(Additive White Gaussian Noise)

Additive (Noise adds to the signal)

(Noise is termed as additive noise)

$$y(t) = x(t) + n(t)$$

$y(t)$ → Received Signal

$x(t)$ → Transmitted Signal

$n(t)$ → Noise

$N(t)$ → Random Process

Popular Noise Model → Gaussian Noise

⇒ Noise is a Gaussian Random Process.

Gaussian Random Process

$N(t)$ is a Gaussian random process if statistics of all orders are jointly Gaussian.

This means :

Considere noise samples

$$\underbrace{N(t_1),\ N(t_2),\ \cdots,\ N(t_k)}_{\text{k - noise samples at time } t_1, t_2, \cdots, t_k.}$$

$$\underbrace{F_{N(t_1), N(t_2) \cdots, N(t_k)}(n_1, n_2, \cdots n_k)}_{\text{Joint distribution of the noise samples.}}$$

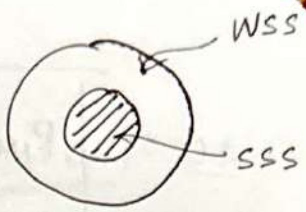
If this is jointly Gaussian; ie follows a multivariate Gaussian Density

For all $t_1, t_2, \cdots t_k$ & for all k.

Then termed as a Gaussian Random Process.

→ Noise + Gaussian Random Process = Gaussian Noise

Gaussian Noise Process



↳ If WSS (Wide Sense Stationary)

Also SSS (Strict Sense Stationary)

All SSS are WSS, converse is not true.

Only for a Gaussian random process, WSS $\Rightarrow$ SSS

$N(t)$ is WSS if

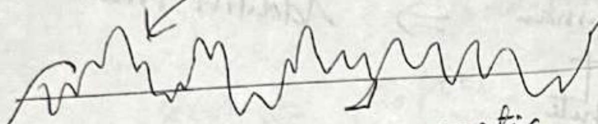
i) $E\{N(t)\} = \mu \quad \forall t$

ii) $E\{N(t)N(t+\tau)\} = R_{NN}(\tau) \quad \forall t, \tau$

↳ depends only on τ (time difference)

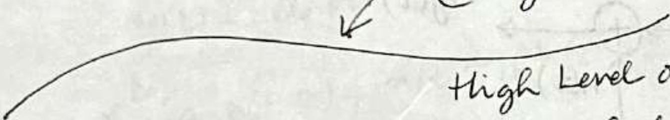
White Noise

Noise typically has very low correlation.



Noise has very low correlation

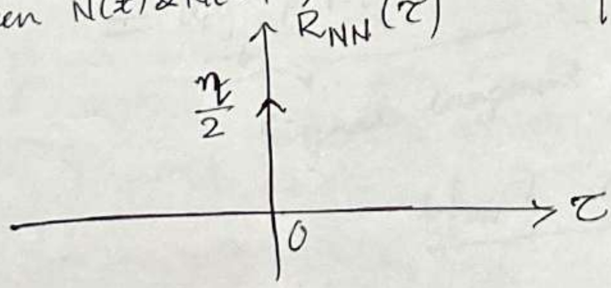
Typical signal (very smooth)



High Level of temporal correlation.

$$\underbrace{E\{N(t)\cdot N(t+\tau)\}}_{\text{Correlation between } N(t)\,\&\,N(t+\tau)} = \frac{\eta}{2}\,\delta(\tau) = R_{NN}(\tau)$$

Correlation $= 0$, if $\tau \neq 0$

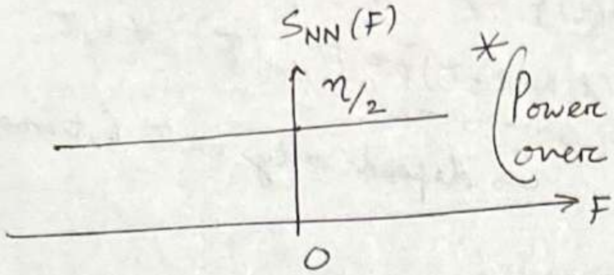


A·CF: $$\boxed{R_{NN}(\tau) = \frac{\eta}{2}\,\delta(\tau)}$$

→ White Noise

Noise samples at any two different time instants t & $t+\tau$ are uncorrelated.

PSD: $$S_{NN}(F) = FT\left[R_{NN}(\tau)\right] = \frac{\eta}{2}$$



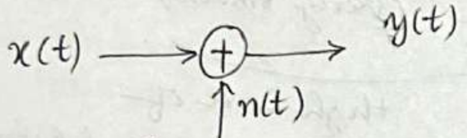
* (Power is spread uniformly over all frequencies) → (Similar to White Light)

↓

'PSD is Flat'

Hence termed as White Noise.

Noise < Additive + Gaussian + White ⇒ Additive White Gaussian Noise

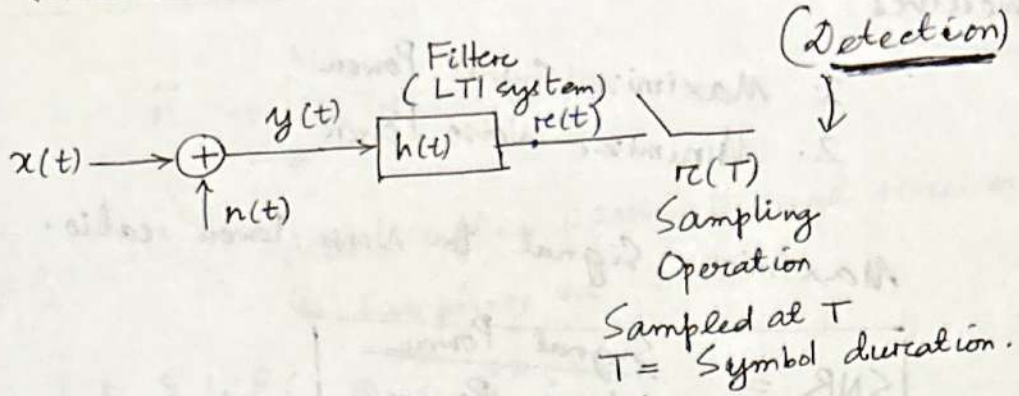


If $n(t)$ is White & Gaussian

⇒ $\boxed{\text{AWGN Channel}}$

Most popular & practically applicable for Digital Communication system.

Digital Communication Receiver



(Schematic diagram of digital comm. receiver)

$$r(t) = y(t) * h(t)$$ ↳ impulse response of LTI s/s (or) Receiver Filter.

$$= \int_{-\infty}^{\infty} y(t-\tau) \cdot h(\tau)\, d\tau$$

$$y(t) = \underbrace{x(t) + n(t)}_{\text{Additive Noise Channel}}$$

$$x(t) = a_0 \, p(t)$$

a_0 → Symbol, $p(t)$ → Pulse

$$y(t) = a_0 \, p(t) + n(t)$$

$$r(t) = \int_{-\infty}^{\infty} \{a_0 \, p(t-\tau) + n(t-\tau)\} \, h(\tau)\, d\tau$$

↳ Sampled at T.

$$r(T) = \int_{-\infty}^{\infty} [a_0 \, p(T-\tau) + n(T-\tau)] \, h(\tau)\, d\tau$$

$$= \underbrace{a_0 \int_{-\infty}^{\infty} p(T-\tau)\, h(\tau)\, d\tau}_{\text{Signal component}} + \underbrace{\int_{-\infty}^{\infty} n(T-\tau)\, h(\tau)\, d\tau}_{\text{Noise component.}}$$

How to design, $h(\tau)$?

Objectives

1. Maximize Signal Power
2. Minimize Noise Power

Maximize Signal-to-Noise Power ratio.

$$\boxed{SNR = \frac{\text{Signal Power}}{\text{Noise Power}}}$$

Fundamental quantity for analysis of performance of a communi-cation system.

Signal:

$$a_0 \int_{-\infty}^{\infty} p(T-\tau) \cdot h(\tau)\, d\tau$$

$$\text{Signal Power} = E\left\{\left|a_0 \int_{-\infty}^{\infty} p(T-\tau) h(\tau)\, d\tau\right|^2\right\}$$

$$= E\left\{|a_0|^2\right\} E\left\{\left|\int_{-\infty}^{\infty} p(T-\tau) h(\tau)\, d\tau\right|^2\right\}$$

$$= \underbrace{E\left\{a_0^2\right\}}_{\text{Symbols are random}} E\left\{\left(\int_{-\infty}^{\infty} p(T-\tau) h(\tau)\, d\tau\right)^2\right\}$$

P_d = Power of Data Symbols

$$P_d = E\left\{|a_0|^2\right\}$$

$\because$ $p(t)$ & $h(t)$ are deterministic

$$= \underbrace{P_d \left(\int_{-\infty}^{\infty} p(T-\tau) h(\tau)\, d\tau\right)^2}_{\text{Signal Power}}$$

Noise Power

$$\tilde{n} = \int_{-\infty}^{\infty} n(T-\tau)\, h(\tau)\, d\tau$$

Noise after passing through receiver filter & sampling at $t = T$.

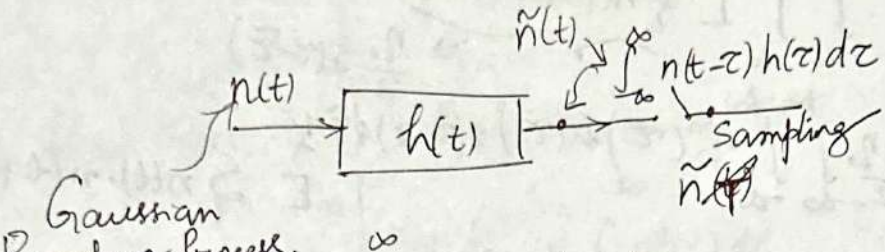
$$E\{|\tilde{n}|^2\} = E\{\tilde{n}^2\}$$

(Assuming all quantities are real)

$n(t) \leftarrow$ White + Gaussian

$$E\{n(t)\} = 0 \quad \text{(Zero mean)}$$

$$E\{n(t) \cdot n(t+\tau)\} = \frac{\eta_0}{2}\delta(\tau) = R_{NN}(\tau)$$



$$\tilde{n}(t) = \int_{-\infty}^{\infty} n(t-\tau)\, h(\tau)\, d\tau$$

Gaussian Random process, since the filter is a Linear system.

$$\tilde{n} = \int_{-\infty}^{\infty} n(T-\tau)\, h(\tau)\, d\tau$$

Gaussian

$$E\{\tilde{n}\} = E\left\{\int_{-\infty}^{\infty} n(T-\tau)\, h(\tau)\, d\tau\right\}$$

$$= \int_{-\infty}^{\infty} \underbrace{E\{n(t-\tau)\}}_{=0} \; \underbrace{h(\tau)}_{\text{Deterministic quantity}}\, d\tau$$

~~since n(t) is~~

$$\therefore \boxed{E\{\tilde{n}\} = 0}$$

$$E\{\tilde{n}^2\} = ?$$

$\hookrightarrow$ Variance or Noise Power

$$= E\{\tilde{n}\cdot\tilde{n}\}$$

$$= E\Big\{\int_{-\infty}^{\infty} n(T-\tau)h(\tau)\,d\tau \times \underbrace{\int_{-\infty}^{\infty} n(T-\tilde{\tau})h(\tilde{\tau})\,d\tilde{\tau}}_{\tilde{n}}\Big\}$$

$$= E\Big\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(T-\tau)\,n(T-\tilde{\tau})\,h(\tau)h(\tilde{\tau})\,d\tau\cdot d\tilde{\tau}\Big\}$$

(Putting the integrals together)

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{E\{n(T-\tau)\cdot n(T-\tilde{\tau})\}}_{\to \frac{\eta_0}{2}\delta(\tau-\tilde{\tau})}\cdot h(\tau)\cdot h(\tilde{\tau})\,d\tau\,d\tilde{\tau}$$

$$= \frac{\eta_0}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)h(\tilde{\tau})\,\delta(\tilde{\tau}-\tau)\,d\tilde{\tau}\,d\tau$$

$$E\{n(t)\cdot n(t+\tau)\} = \frac{\eta_0}{2}\delta(\tau)$$

Depends on Time difference τ

$$= \frac{\eta_0}{2}\int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} h(\tilde{\tau})\,\delta(\tilde{\tau}-\tau)\,d\tilde{\tau}\right)d\tau$$

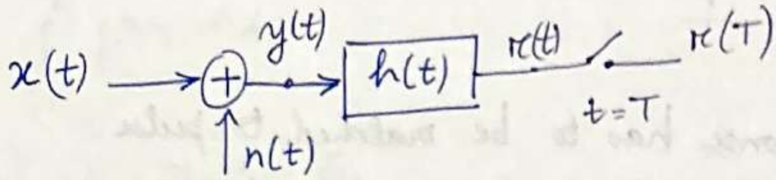
$$= \frac{\eta_0}{2}\int_{-\infty}^{\infty} h(\tau)\cdot h(\tau)\,d\tau$$

$$= \frac{\eta_0}{2}\int_{-\infty}^{\infty} |h(\tau)|^2\,d\tau = \frac{\eta_0}{2}E_h \quad \{E_h = \text{Energy of Filter}\}$$

$$\boxed{E\{\tilde{n}^2\} = \frac{\eta_0}{2}E_h = \frac{\eta_0}{2}\int_{-\infty}^{\infty}|h(\tau)|^2\,d\tau}$$

$\hookrightarrow$ Noise Power at o/p of sampler.

Or Variance of $\tilde{n}$



Signal $\rightarrow \int_{-\infty}^{\infty} a_0\, p(T-\tau)\, h(\tau)\, d\tau$

Noise $\rightarrow \int_{-\infty}^{\infty} n(T-\tau)\, h(\tau)\, d\tau$

$$\text{SNR} = \frac{\text{Signal Power}}{\text{Noise Power}}$$

<u>Signal-to-Noise Power Ratio</u>

$$= \frac{P_d \left(\int_{-\infty}^{\infty} p(T-\tau)\, h(\tau)\, d\tau \right)^2}{\frac{\eta_0}{2} \int_{-\infty}^{\infty} |h(\tau)|^2\, d\tau}$$

We have to find filter $h(t)$ which maximizes SNR

$$\int_{-\infty}^{\infty} (u(t)\, v(t))^2\, dt \leq \int_{-\infty}^{\infty} u^2(t)\, dt \times \int_{-\infty}^{\infty} v^2(t)\, dt$$

<u>Cauchy-Schwarz inequality</u>

Equality holds when

$u(t) \propto v(t)$

$\Rightarrow u(t) = k\, v(t)$

Using Cauchy-Schwarz inequality,

$$\text{SNR} \leq \frac{P_d \int_{-\infty}^{\infty} p^2(T-\tau)\, d\tau \times \int_{-\infty}^{\infty} h^2(\tau)\, d\tau}{\frac{\eta_0}{2} \int_{-\infty}^{\infty} h^2(\tau)\, d\tau}$$

$$\Rightarrow \text{SNR} \leq \frac{P_d \int_{-\infty}^{\infty} p^2(T-\tau)\, d\tau}{\left(\frac{\eta_0}{2}\right)}$$

SNR is maximized when Equality holds ~~when~~ &

$$p(T-\tau) \propto h(\tau)$$

$$\Rightarrow p(T-\tau) = k\, h(\tau)$$

Without loss of generality $k=1$

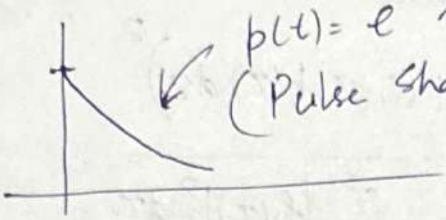
$$\boxed{h(\tau) = p(T-\tau)}$$

Principle of Matched Filter

impulse response has to be matched to pulse shaping filter. Hence it is termed as a matched filter.

To maximize SNR at receiver output, one has to employ a matched filter.

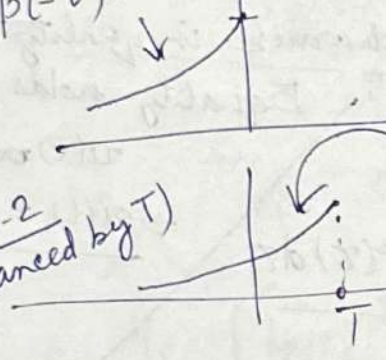
$p(t) = e^{-t} u(t)$ (Pulse Shaping Filter)



$$h(\tau) = p(T-\tau)$$

Step - 1 (Flip)

$p(-t)$



$p(T-\tau) = h(\tau)$

Step-2 (Advanced by T)

Maximum SNR for the matched Filter is:-

$$SNR = \frac{P_d \int_{-\infty}^{\infty} p^2(T-\tau)\, d\tau}{\left(\frac{\eta_0}{2}\right)}$$

$$= \frac{P_d \int_{-\infty}^{\infty} |h^2(\tau)|\, d\tau}{\frac{\eta_0}{2}}$$

$$\int_{-\infty}^{\infty} p^2(T-\tau)\,d\tau$$

$$T-\tau = \tau'$$

$$\Rightarrow -d\tau = d\tau'$$

$$\int_{\infty}^{-\infty} p^2(\tau')\,(-d\tau')$$

$$= \underbrace{\int_{-\infty}^{\infty} p^2(\tau)\,d\tau}_{\text{Energy of pulse shaping filter}}$$

$$\boxed{\begin{aligned} SNR &= \frac{P_d}{\left(\frac{\eta_0}{2}\right)} \int_{-\infty}^{\infty} h^2(\tau)\,d\tau \\ &= \frac{P_d}{\left(\frac{\eta_0}{2}\right)} \int_{-\infty}^{\infty} p^2(\tau)\,d\tau \end{aligned}}$$

Maximum SNR for matched Filter

Probability of Error in Digital Comm. System

→ Transmit several symbols

→ The Prob. of error is an important metric to characterize the performance of a digital communication system.

→ Aim is to minimize the Prob of Error

$(\sim 10^{-6} - \sim 10^{-8})$

Typical values of Prob of Error.

For a given scheme, how to characterize the prob. of error?

After filtering with $h(t)$ & ~~the~~ sampling at $t = T$

$$r(T) = a_0 \int_{-\infty}^{\infty} p(T-\tau)\, h(\tau)\, d\tau + \int_{-\infty}^{\infty} n(T-\tau)\, h(\tau)\, d\tau$$

$$= a_0 \int_{-\infty}^{\infty} p(T-\tau)\, h(\tau)\, d\tau + \tilde{n}$$

Gaussian

To maximize SNR

$$\boxed{p(T-\tau) = h(\tau)}$$

"Matched Filter"

$$r(T) = a_0 \underbrace{\int_{-\infty}^{\infty} p^2(T-\tau)\, d\tau}_{= \int_{-\infty}^{\infty} p^2(\tau)\, d\tau \,=\, E_p} + \tilde{n} = a_0 E_p + \tilde{n}$$

Energy of pulse shaping filter

$$r(T) = \underbrace{a_0}_{\text{Transmitted Symbol}} \underbrace{E_p}_{\text{Pulse Energy}} + \underbrace{\tilde{n}}_{\text{Gaussian Noise}}$$

$E\{\tilde{n}\} = 0$ (Zero mean)

$$E\{\tilde{n}^2\} = \frac{\eta_0}{2} \int_{-\infty}^{\infty} |h(\tau)|^2 d\tau$$

$$= \frac{\eta_0}{2} \int_{-\infty}^{\infty} p^2(\tau) d\tau = \frac{\eta_0}{2} E_p$$

$$\boxed{\underbrace{\tilde{n}}_{\text{(Noise after Sampling)}} \sim \underbrace{\mathcal{N}}_{\text{Gaussian}}\left(\underbrace{0}_{\text{Mean}}, \underbrace{\frac{\eta_0}{2} E_p}_{\text{Variance}}\right)}$$

$\mathcal{N}(\mu, \sigma^2)$

↳ Gaussian of

~~Noise~~ mean $= \mu$

Variance $= \sigma^2$

$\mathcal{N}(\underbrace{0}_{\text{Mean}}, \underbrace{1}_{\text{Variance = 1 (unity)}})$

Standard Gaussian

(or) Standard Normal

$r(T) = a_0 E_p + \tilde{n}$

↳ Symbols must belong to a digital Constellation in a digital Comm. System.

$a_0 = -A$ or $+A$

$$\begin{array}{cccccc} 0 & 1 & 1 & 0 & 1 & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ A, & -A, & -A, & A, & -A, & A \end{array} \cdots \quad \text{Information bits}$$

$$\left.\begin{array}{l} 0 \rightarrow A \\ 1 \rightarrow -A \end{array}\right\} \text{Mapping}$$

$$r(T) = \pm A E_p + \tilde{n}$$

$$r(T) = A E_p + \tilde{n}, \quad a_0 = A$$

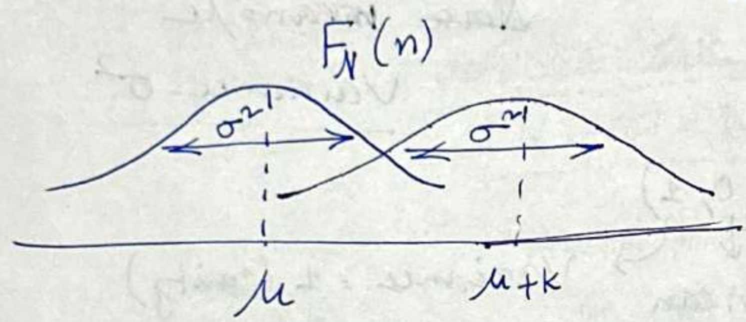
$$r(T) = -A E_p + \tilde{n}, \quad a_0 = -A$$

O/p after sampling

<u>Property</u> :

If $n \sim \mathcal{N}(\mu, \sigma^2)$

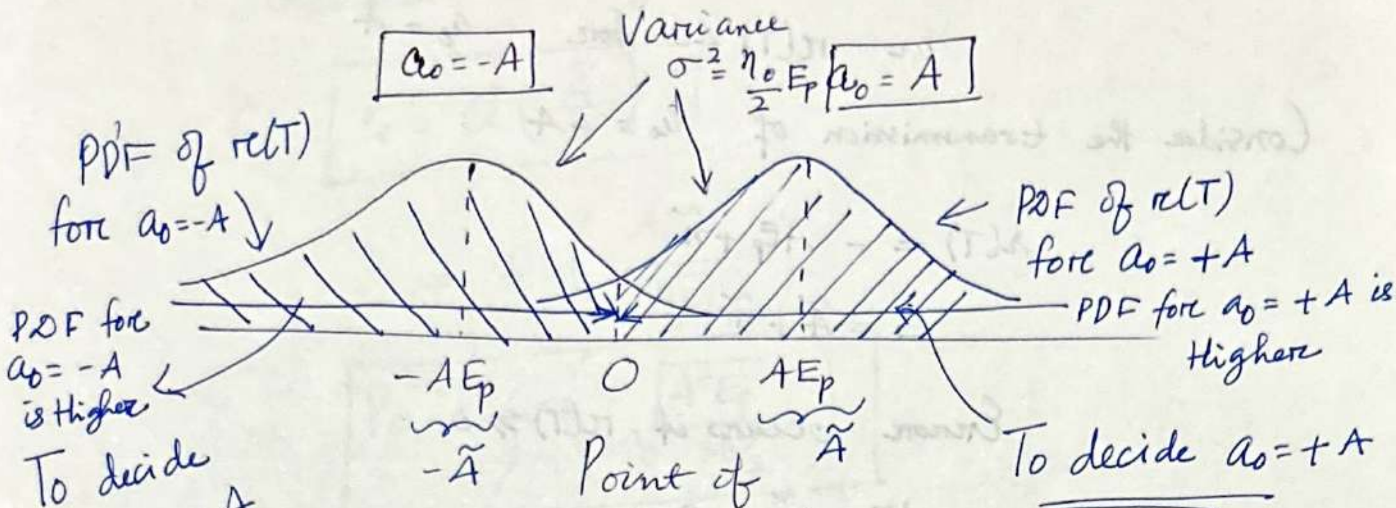
Then $\underbrace{n+k}_{\text{constant}} \sim \mathcal{N}\big((\mu+k), \underbrace{\sigma^2}_{\text{Variance is unchanged.}}\big)$



$$r(T) = \begin{cases} A E_p + \tilde{n} \sim \mathcal{N}(A E_p, \sigma^2) \\ -A E_p + \tilde{n} \sim \mathcal{N}(-A E_p, \sigma^2) \end{cases}$$

$r(T)$ is Gaussian, Mean $= AE_p$ | $r(t)$ is Gaussian, Mean $= -AE_p$.

For Both cases variance is same, $\sigma^2 = \cancel{A}\,\frac{\eta_0}{2} E_p$.



Point of intersection is (0 by symmetry)

Both have same variance

$$r(T) = \begin{cases} AE_p + \tilde{n} \rightarrow \mathcal{N}\left(AE_p, \frac{N_0}{2}E_p\right) \\ -AE_p + \tilde{n} \rightarrow \mathcal{N}\left(-AE_p, \frac{N_0}{2}E_p\right) \end{cases}$$

$\tilde{n} \rightarrow \cancel{\mathcal{N}(0, E_p)}$ $\mathcal{N}\left(0, \frac{N_0}{2}E_p\right)$

$N_0 = \frac{\eta_0}{2}$

$$r(T): \begin{cases} \mathcal{N}(\tilde{A}, \sigma^2), \text{ for } a_0 = +A \\ \mathcal{N}(-\tilde{A}, \sigma^2), \text{ for } a_0 = -A \end{cases}$$

Detection rule / Decision rule

$$\begin{cases} r(T) \geq 0 \Rightarrow \text{decide } a_0 = A \\ r(T) < 0 \Rightarrow \text{decide } a_0 = -A \end{cases}$$

Corresponds to information bit 0

Corresponds to information bit 1

Probability of Error

When does error occur?

Error occurs when

$$r(T) \geq 0 \quad \text{for} \quad a_0 = -A$$

$$\text{or} \quad r(T) < 0 \quad \text{for} \quad a_0 = A$$

Consider the transmission of $a_0 = -A$

$$r(T) = -AE_p + \tilde{n}$$

$$= -\tilde{A} + \tilde{n}$$

Error occurs if $r(T) \geq 0$

$$\Rightarrow -AE_p + \tilde{n} \geq 0$$

$$\Rightarrow \tilde{n} \geq AE_p = \tilde{A}$$

$$P_e = P_r(\tilde{n} \geq \tilde{A})$$

$$F_N(\tilde{n}) = \frac{1}{2\pi\sigma^2} e^{-\frac{\tilde{n}^2}{2\sigma^2}}$$

$$= \int_{\tilde{A}}^{\infty} F_N(\tilde{n})\, d\tilde{n}$$

$$= \int_{\tilde{A}}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tilde{n}^2}{2\sigma^2}}\, d\tilde{n}$$

$$\frac{\tilde{n}}{\sigma} = n' \Rightarrow d\tilde{n} = \sigma\, dn'$$

$$= \int_{\frac{\tilde{A}}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi\cancel{\sigma^2}}} e^{-\frac{n'^2}{2}} \cancel{\sigma}\, dn'$$

$$= \int_{\frac{\tilde{A}}{\sigma}}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{n'^2}{2}}}_{\text{PDF of standard Gaussian with mean}=0 \text{ \& variance}=1} dn'$$

$$Q(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

Gaussian Q function denotes that standard Gaussian RV is greater than u.

So, $\boxed{P_e = Q\left(\frac{\tilde{A}}{\sigma}\right)}$

$$= Q\left(\frac{AE_p}{\sqrt{\frac{N_0}{2}E_p}}\right)$$

$$\boxed{P_e = Q\left(\sqrt{\frac{A^2 E_p}{N_0/2}}\right)}$$

Prob. of Error

Gaussian Q function.

BINARY PHASE SHIFT KEYING (BPSK)

– A digital modulation scheme.

BPSK

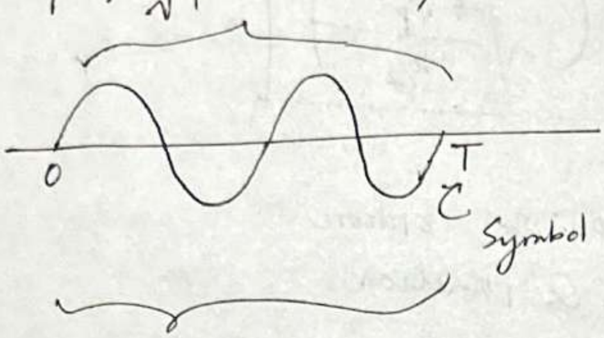
Pulse, $p(t) = \sqrt{\frac{2}{T}} \cos(2\pi F_c t)$, $0 \le t \le T$

$T \downarrow$ Symbol duration

T = Integral multiple of $\frac{1}{F_c}$

$\Rightarrow \boxed{T = \frac{k}{F_c}}$

$p(t) = \frac{\sqrt{2}}{\sqrt{T}} \cos(2\pi F_c t)$



↳ Contains 2 cycles of cosine waveform.

$\Rightarrow \boxed{T = \frac{2}{F_c}}$

$p(t)$ is normalized to unit energy

$$E_p = \int_{-\infty}^{\infty} p^2(t)\,dt = \int_0^T \frac{2}{T}\cos^2(2\pi F_c t)\,dt$$

$$= \frac{2}{T}\int_0^T \frac{1 + \cos 4\pi F_c t}{2}\,dt = \frac{2}{T} \times \frac{1}{2} T = 1$$

$\Rightarrow \boxed{E_p = 1}$

$x(t) = a_0\, p(t)$, $a_0 \in \{-A, +A\}$

$-A \uparrow 1$, $+A \uparrow 0$

Each a_0 carries 1 bit of information.

E_b = Energy per bit

Constant across schemes.

information bit $\begin{cases} 0 \rightarrow \text{occurs with prob } \frac{1}{2} \\ 1 \rightarrow \quad " \qquad " \quad 1/2 \end{cases}$

$a_0 = \pm A \qquad x(t) = a_0 p(t)$

Average Energy per bit

$$\frac{1}{2} A^2 E_p + \frac{1}{2} A^2 E_p = A^2 E_p = A^2 \quad (\because E_p = 1)$$

$$A^2 = E_b$$

$$\Rightarrow \boxed{A = \sqrt{E_b}}$$

This ensures average ~~f~~ energy/bit $= E_b$

Therefore, the transmitted waveforms are,

$$0 \rightarrow a_0 = +A = \sqrt{E_b}$$

$$\therefore x(t) = \sqrt{E_b} \sqrt{\frac{2}{T}} \cos(2\pi F_c t), \quad 0 \le t \le T$$

$$\boxed{x(t) = \sqrt{\frac{2E_b}{T}} \cos(2\pi F_c t)}, \quad 0 \le t \le T$$

waveform corresponding to bit $\underline{0}$

$$1 \rightarrow a_0 = -A \triangleq -\sqrt{E_b}$$

$$\boxed{x(t) = -\sqrt{E_b} \sqrt{\frac{2}{T}} \cos(2\pi F_c t)}, \quad 0 \le t \le T$$

waveform corresponding to bit 1

Waveforms

0 $\swarrow$ $\searrow$ 1

$\sqrt{\frac{2E_b}{T}}\cos(2\pi F_c t)$ $\qquad$ $-\sqrt{\frac{2E_b}{T}}\cos(2\pi F_c t)$

$= \sqrt{\frac{2E_b}{T}}\cos(2\pi F_c t + \pi)$

Shift by π

Shift by π

$$\sqrt{\frac{2E_b}{T}}\cos\left(2\pi F_c t + \pi + \pi\right) = \sqrt{\frac{2E_b}{T}}\cos(2\pi F_c t)$$

Two waveforms $\Rightarrow$ Binary modulation scheme

+

Waveforms are phase shifted w.r.t. each other.

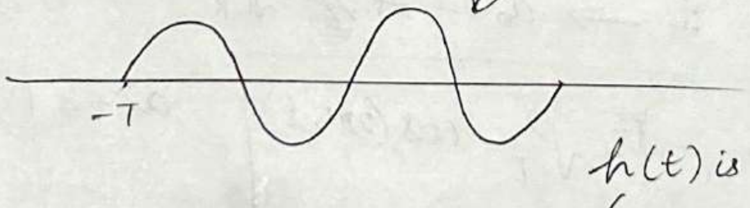
BPSK $\to$ ~~β~~ $A\,p(t)$, $\;-A\,p(t)$

$p(t) = \sqrt{\frac{2}{T}}\cos(2\pi F_c t)$

$A = \sqrt{E_b}$

At receiver, matched filter is $h(t) = p(T-\tau)$

$p(-t)$ is from $-T$ to 0



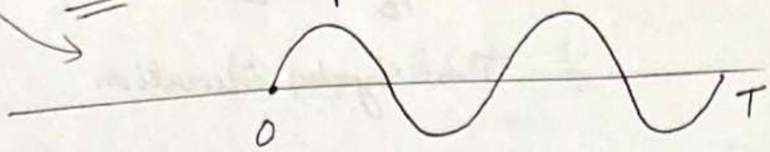
$h(t)$ is nonzero for $t \leq 0$

$\Downarrow$

Non-causal system.

Shifting by T, to make it causal.
(Delaying)

Causal $p(T-t)$



$r(T) \geq 0 \Rightarrow$ decide $a_0 = A = \sqrt{E_b} \Rightarrow$ bit $= 0$

$r(T) < 0 \Rightarrow$ decide $a_0 = -A = -\sqrt{E_b} \Rightarrow$ bit $= 1$

optimal decision rule

$$P_e = Q\left(\sqrt{\frac{A^2 E_p}{N_0/2}}\right)$$

$$A = \sqrt{E_b}$$
$$E_p = 1$$

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

Prob. of error of BPSK with avg. energy per bit $= E_b$

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$ → Denotes the prob.

that $\mathcal{N}(0,1) \geq x$

AMPLITUDE ~~PHA~~ SHIFT KEYING (ASK)

→ A digital modulation scheme.

$$p(t) = \sqrt{\frac{2}{T}}\cos(2\pi F_c t),\ 0 \le t < T$$

$$T = \frac{k}{F_c},\ \underline{k \in Z}$$

$T \rightarrow$ Symbol Duration

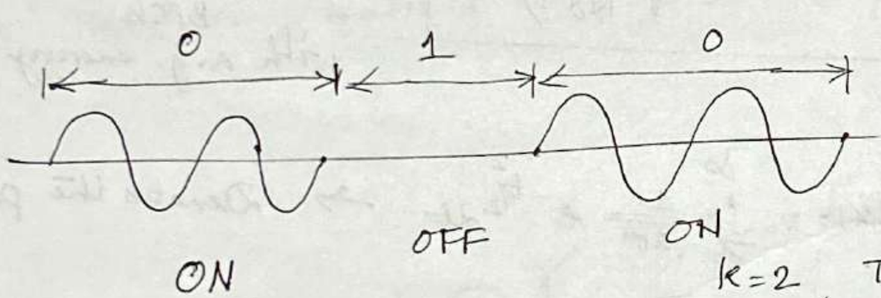
$$E_p = \int_0^T p^2(t)\,dt = 1$$

Energy of pulse is unity

$$a_0 = \{0, A\} \Rightarrow a_0 = A \text{ or } 0$$

waveforms differ in amplitude.

information bits
$$\begin{cases} 0 \longrightarrow A\sqrt{\frac{2}{T}}\cos(2\pi F_c t), & 0 \le t < T \\ 1 \longrightarrow 0 \cdot p(t) = 0, & 0 \le t \le T \end{cases}$$



$k = 2,\ T = \frac{2}{F_c}$

ASK is also termed as ON-OFF Keying.

How to chose A?

Bit energy = E_b (constant)

$$x(t) \begin{cases} \xrightarrow{0} A\,p(t), & \text{if } a_0 = 0 \\ \xrightarrow{1} 0, & \text{if } a_0 = 1 \end{cases}$$

$$P_r(0) = P_r(1) = \tfrac{1}{2}$$

Average energy per bit

$$= \frac{1}{2}A^2E_p + 0 = \frac{1}{2}A^2E_p.$$

$$\frac{1}{2}A^2E_p = E_b$$

$$\Rightarrow \frac{A^2}{2} = E_b \quad (\because E_p = 1)$$

$$\Rightarrow \boxed{A = \sqrt{2E_b}}$$

→ This ensures energy/bit is constant across the schemes.

$0 \rightarrow A\,p(t)$

$1 \rightarrow 0\,p(t)$

both waveforms differ in amplitude

⇒ This modulation scheme is termed as amplitude shift keying (ASK).

Prob. of error,

$$x(t) \begin{cases} \xrightarrow{0} A\,p(t) \\ \xrightarrow{1} 0 \end{cases}$$

At the receiver

$$y(t) = x(t) + \underbrace{n(t)}_{\text{AWGN.}}$$

Matched Filter with

$$h(t) = p(T-t)$$

Followed by sampling at $t = T$.

Consider, transmission bit $= 0$

$$y(t) = Ap(t) + n(t)$$

After matched filtering followed by sampling at $t = T$,

$$r(T) = A\,E_p + \tilde{n}$$

Gaussian

mean $= 0$

Variance $= \frac{N_0}{2} E_p = \frac{N_0}{2}$

Consider, information bit $= 1$

$$y(t) = 0 \cdot p(t) + n(t) = n(t)$$

After matched filtering & sampling at $t = T$.

$$r(T) = \tilde{n}$$

Gaussian

Mean $= 0$

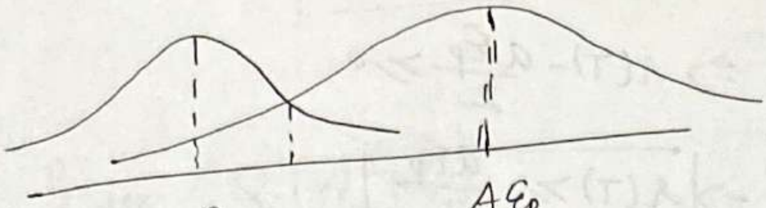
Variance $= \frac{N_0}{2} E_p$

$$r(T) = \begin{cases} A\mathcal{E}_p + \tilde{n}, & \text{if } a_0 = A \\ 0 + \tilde{n}, & \text{if } a_0 = 0 \end{cases}$$

What is the optimal decision rule?

$r(T)$ is Gaussian with mean $= A\mathcal{E}_p$

Gaussian with mean $= 0$



By Symmetry, midpoint $= \frac{A\mathcal{E}_p}{2}$

Optimum decision rule:

$$\boxed{\begin{aligned} r(T) &\geq \frac{A\mathcal{E}_p}{2} \Rightarrow a_0 = A \\ r(T) &< \frac{A\mathcal{E}_p}{2} \Rightarrow a_0 = 0 \end{aligned}}$$

Optimum decision rule.

$$\tilde{r}(T) = r(T) - \frac{A\mathcal{E}_p}{2} = \begin{cases} \frac{A\mathcal{E}_p}{2} + \tilde{n}, & \text{if } a_0 = A \\ -\frac{A\mathcal{E}_p}{2} + \tilde{n}, & \text{if } a_0 = 0 \end{cases}$$

Similar to BPSK with $\cancel{A}\mathcal{E}_p$ replaced by $\frac{A\mathcal{E}_p}{2}$

Decide

$$~~a_0 = A \quad \text{if} \quad \tilde{r}(T) \geq 0 \Rightarrow r(T) \geq \frac{A\mathcal{E}_p}{2}~~$$

$$~~a_0 = 0 \quad \text{if} \quad \tilde{r}(T) < 0 \Rightarrow r(T) < \frac{A\mathcal{E}_p}{2}~~$$

Decide $a_0 = A$.

if $\tilde{r}(T) \geq 0$

$$\Rightarrow r(T) - \frac{a\mathcal{E}_p}{2} \geq 0$$

$$\Rightarrow \boxed{r(T) \geq \frac{a\mathcal{E}_p}{2}}$$

$r(T) \geq \frac{a\mathcal{E}_p}{2} \Rightarrow$ Decide $a_0 = A$

$r(T) < \frac{a\mathcal{E}_p}{2} \Rightarrow$ Decide $a_0 = -A$

$$\tilde{r}(T) = \begin{cases} \frac{A\mathcal{E}_p}{2} + \tilde{n} \\ -\frac{A\mathcal{E}_p}{2} + \tilde{n} \end{cases}$$

Prob. of bit error, is similar to BPSK with $A\mathcal{E}_p$ replaced by $\frac{A\mathcal{E}_p}{2}$

$$P_e = Q\left(\frac{A\mathcal{E}_p/2}{\sqrt{\frac{N_0}{2}\mathcal{E}_p}}\right) = Q\left(\sqrt{\frac{A^2\mathcal{E}_p}{2N_0}}\right)$$

$\mathcal{E}_p = 1$, $A^2 = 2\mathcal{E}_b$

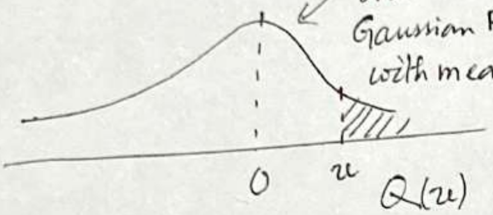
$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{2N_0}}\right) = Q\left(\frac{E_b}{N_0}\right)}$$

Probability of error for ASK

for BPSK

$$\boxed{P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

$$\begin{cases} P_{e,\,ASK} = Q\left(\sqrt{E_b/N_0}\right) \\ P_{e,\,BPSK} = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \end{cases}$$



Standard Gaussian RV with mean = 0 & var = 1

$Q(x)$

Prob that Standard Gaussian $\geq x$

$$= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$Q\left(\sqrt{\frac{2E_b}{N_0}}\right) < Q\left(\sqrt{E_b/N_0}\right)$$

$$\Rightarrow P_{e,\,BPSK} \leq P_{e,\,ASK}$$